

# Electromagnetic Field Theory without Divergence Problems 2. A Least Invasively Quantized Theory<sup>1, 2</sup>

Michael K.-H. Kiessling<sup>3</sup>

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Classical electrodynamics based on the Maxwell–Born–Infeld field equations coupled with a Hamilton–Jacobi law of point charge motion is partially quantized. The Hamilton–Jacobi phase function is supplemented by a dynamical amplitude field on configuration space. Both together combine into a single complex wave function satisfying a relativistic Klein–Gordon equation that is self-consistently coupled to the evolution equations for the point charges and the electromagnetic fields. Radiation-free stationary states exist. The hydrogen spectrum is discussed in some detail. Upper bounds for Born’s “aether constant” are obtained. In the limit of small velocities of and negligible radiation from the point charges, the model reduces to Schrödinger’s equation with Coulomb Hamiltonian, coupled with the de Broglie–Bohm guiding equation.

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**KEY WORDS:** *Spacetime:* special relativity, space-like foliations; *Electromagnetism:* electromagnetic fields, point charges, wave functions; *Determinism:* Maxwell–Born–Infeld field equations, de Broglie–Bohm law of quantum motion, Klein–Gordon equation; *Probability:* configuration space, Born’s statistical law.

## 1. INTRODUCTION

In our previous paper<sup>(29)</sup> we presented the first relativistic classical electromagnetic field theory in which the notion of the (spin-less) point electron is satisfactorily implemented. The classical theory formulated in ref. 29 is divergence problem-free, *a priori* speaking; in particular, no regularization or renormalization is needed to give sense to the basic variables of the

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<sup>2</sup> Part II of two parts in honor of Elliott H. Lieb’s 70th birthday.

<sup>3</sup> Department of Mathematics, Rutgers, The State University of New Jersey, 110 Frelinghuysen Rd., Piscataway, New Jersey 08854; e-mail: miki@math.rutgers.edu

theory. Half of this feat was actually accomplished long ago,<sup>4</sup> by Born and Infeld.<sup>(14)</sup> Their nonlinear Maxwell–Born–Infeld field equations eliminate the infinite classical electromagnetic self-energy problem for point charges, and since refs. 7 and 31 we know that they do so in a compellingly unique way. Unfortunately, although no longer diverging, the Lorentz self-force remained ill-defined in magnitude and direction. It is not the case, as the founding fathers of that theory believed, that this Lorentz self-force problem could be overcome by simply regularizing and then taking limits, and/or imposing energy conservation. Thus, contrary to Dirac’s early proclamation that “[t]he classical theory is found to be completely satisfactory” (ref. 19, p. 32), a satisfactory law of motion for the point charges has in fact been missing. Our main contribution in ref. 29 is to supply a well-defined law of point charge motion. A relativistic many-body Hamilton–Jacobi equation has to be solved together with a system of Maxwell–Born–Infeld field equations for generic point charge sources; i.e., instead of the actual Maxwell–Born–Infeld fields one studies a whole family of such fields indexed by their generic point sources configuration. The Hamilton–Jacobi guiding law is solved subsequently to get the actual particles’ motions, and when this actual point charges configuration is substituted for the generic configuration in the indexed fields, the actual electromagnetic Maxwell–Born–Infeld fields are obtained.

This classical electromagnetic theory with point electrons is in itself an interesting object for further study, but the really interesting question is whether it can serve as stepping stone en route to a consistent quantum theory of electromagnetism with point electrons. For Born and Infeld,<sup>(9–11, 13–15)</sup> Pryce,<sup>(32–35)</sup> Schrödinger,<sup>(37–40)</sup> and Dirac,<sup>(19)</sup> this was the driving force behind their quest, but the attempts in refs. 15, 19, and 35 to quantize the Maxwell–Born–Infeld field equations revealed that “difficulties arise with the passage to the quantum theory, which appear to be insoluble with present methods of quantization” (ref. 19, p. 32), and “[t]he adaption of these ideas to the principles of quantum theory and the introduction of the spin has [...] met with no success” (ref. 12, p. 375). One reason for the failure of these attempts is of course the fact that the Maxwell–Born–Infeld field equations with point charges do not in themselves constitute a complete classical dynamical theory, but this is not the only reason. The other, not less important reason is their choice of quantization procedure, which was patterned after the available standard procedure of replacing classical quantities by operators. However, if one wants to use a classical divergence problem-free electromagnetic field theory with point electrons as

<sup>4</sup> Our paper<sup>(29)</sup> contains a fairly exhaustive collection of scientific and bibliographical background information which is also pertinent to the present paper.

point of departure for the construction of a divergence problem-free electromagnetic quantum theory with point electrons, then one can reasonably hope to be successful only if one tries *not to tamper* with the integrity of the mathematical structures which are responsible for the absence of any divergence problems at the classical level. In this spirit we have applied a “least invasive quantization procedure” to the classical electromagnetic field theory developed in our previous paper.<sup>(29)</sup> The electromagnetic quantum theory with point electrons which results from this is the subject of the present paper.

Like the classical theory from which it springs, the quantum theory describes the joint dynamics of spin-less point charges and the total electromagnetic fields. The point charges move according to a relativistic generalization of the first order guiding equation alluded to by Born<sup>5</sup> in ref. 8 and the explicit form of which de Broglie<sup>(16)</sup> and Bohm<sup>(6)</sup> discovered subsequently. The guiding field is the gauge-invariant gradient of the phase of a wave function solving a relativistic Klein–Gordon equation for the electromagnetic potentials of the total electromagnetic fields indexed by generic point sources, which in turn are obtained from the Maxwell–Born–Infeld field equations with generic point charges as sources. The guiding equation is to be solved subsequently to obtain the motion of the actual point charges, and when this actual point charges configuration is substituted for the generic ones in the indexed fields, the actual electromagnetic Maxwell–Born–Infeld fields are obtained. While this *partially* quantized theory is certainly only a modest step forward, for spin and photon are not yet incorporated, the theory is a priori free of any divergence problems; hence, once again there is no need for regularization or, for that matter, renormalization. We take this as a major encouragement to pursue the full quantization, with spin and photon, in due course.

In this paper we also re-address the subtle issue of the value of “Born’s aether constant,” the new dimensionless physical constant that enters the Born–Infeld law of the “aether.”<sup>6</sup> In ref. 29 we found that Born’s reasoning<sup>(9)</sup> that the value of this aether constant be chosen so that the empirical electron rest energy  $m_e c^2$  equals the now finite electrostatic energy of a point charge at rest, is not conclusive at the classical level. In principle the value of the aether constant  $\beta$  should be inferable from the spectral data, but that means its true value will be computable only after spin, and perhaps even the photon, are implemented into the theory. Nevertheless, by discussing the “spin-less hydrogen” spectrum in some detail we here find

<sup>5</sup> Born seems to have favored a stochastic guiding equation but remarked that Frenkel had pointed out the possibility of a deterministic guiding equation.

<sup>6</sup> In our previous paper<sup>(29)</sup> we stipulated that “aether” is short for “electromagnetic vacuum.”

some decent upper bounds on Born's aether constant that, curiously, still leave the value computed in refs. 9 and 13 viable, for now. Incidentally, for our discussion of the hydrogen spectrum we also prove the first rigorous two-body results for the nonlinear Maxwell–Born–Infeld field equations.

In the remainder of this paper, we first present the least invasive quantization of the classical theory, using the compact, manifestly Poincaré- and Weyl-covariant formalism. We will then discuss the spin-less hydrogen spectrum, for which purpose we prove the first rigorous results for the classical Maxwell–Born–Infeld field equations with two point charges. The paper concludes, after a summary, with an outlook and an epilogue in celebration of Elliott H. Lieb's 70th birthday.

## 2. THE ELECTROMAGNETIC QUANTUM THEORY IN COVARIANT FORMAT

### 2.1. The Basic Equations

As in ref. 29 we use dimensionless units with the following conversion factors between Gaussian and dimensionless units:  $\hbar$  (Planck's constant divided by  $2\pi$ ) for both the unit of action and the magnitude of angular momentum,  $e$  (elementary charge) for the unit of charge,  $m_e$  (electron rest mass) for the unit of mass,  $c$  (speed of light *in vacuo*) for the unit of speed. Thus, length and time are both referred to in the same dimensionless unit, multiples of the Compton wave length of the electron  $\lambda_C = \hbar/m_e c$ . Accordingly, the unit magnitude of the electromagnetic fields is to be converted by a factor  $e/\lambda_C^2$ , while the natural unit for the magnitude of momentum and the energy are converted, respectively, by factors  $m_e c$  and  $m_e c^2$ . The parameter  $\alpha$  will denote Sommerfeld's fine structure constant.

#### 2.1.1. The Equations of the Flat Electromagnetic Spacetime

In our partially quantized theory, the (flat) *electromagnetic spacetime* structure is defined as in our classical theory. Thus, Minkowski spacetime  $\mathbb{M}^4$  is made into an electromagnetic spacetime by decorating it with a classical electromagnetic field which satisfies the Maxwell–Born–Infeld field equations in a distributional sense; the field may not be well defined along one-dimensional time-like defects. When cut with a space-like slice, the electromagnetic field is finite in a punctured space-like neighborhood of these line defects, which themselves are noticeable as moving point charges in the space-like slice(s). To have the quantum theory minimally self-contained, we briefly recall these laws in the genuinely electromagnetic setting, in which all point charges are positive or negative unit charges, representing electrons of either variety.

Let  $H_k$  be the point history (future oriented time-like world-line) of the  $k$ th particle, and let  $\cup_k H_k$  denote the set of  $N$  point histories with which  $\mathbb{M}^4$  is threaded. Faraday's electromagnetic field tensor  $\mathbf{F}$  is a two-form on  $\mathbb{M}^4 \setminus \cup_k H_k$  satisfying the *Faraday–Maxwell law*<sup>(30)</sup>  $d\mathbf{F} = \mathbf{0}$  in the sense of distributions. Let  $\star\mathbf{F}$  (etc.) be the Hodge dual of  $\mathbf{F}$  (etc.). Then the *Born and Infeld law of the aether*,

$$-\star\mathbf{M} = \frac{\mathbf{F} - \beta^4 \star(\mathbf{F} \wedge \mathbf{F}) \star \mathbf{F}}{\sqrt{1 - \beta^4 \star(\mathbf{F} \wedge \star\mathbf{F}) - \beta^8 (\star(\mathbf{F} \wedge \mathbf{F}))^2}}, \quad (1)$$

in which  $\beta \in (0, \infty)$  is Born's aether constant, maps  $\mathbf{F}$  to Maxwell's electromagnetic displacement tensor  $\mathbf{M}$ , which is a two-form on  $\mathbb{M}^4 \setminus \cup_k H_k$  satisfying the *Ampère–Coulomb–Maxwell law*<sup>(30)</sup>  $d\mathbf{M} = 4\pi\mathbf{J}$  in the sense of distributions, where (cf. refs. 27, 42, and 45)

$$\mathbf{J}(\varpi) = \sum_{k \in \mathcal{N}} \int_{-\infty}^{+\infty} z_k \star\mathbf{u}_k(\tau) \delta_{\eta_k(\tau)}(\varpi) d\tau, \quad (2)$$

is the electromagnetic current density at  $\varpi \in \mathbb{M}^4$  of a system of  $N \geq 0$  electric unit point charges. Here,  $\delta_{\eta(\tau)}(\cdot)$  is the Dirac measure on  $\mathbb{M}^4$  concentrated at  $\eta(\tau)$ , where  $\tau$  is a Lorentz-scalar time parameter. If  $\tau$  is proper-time, then  $\mathbf{u}_k(\tau)$  is the future-oriented Minkowski-velocity co-vector, which is the metrical dual of the Minkowski-velocity vector  $d\varpi/d\tau|_{\varpi=\eta_k(\tau)}$  of the  $k$ th point charge, and  $\star\mathbf{u}_k(\tau)$  is the Hodge dual of  $\mathbf{u}_k(\tau)$ . Furthermore,  $z_k$  is the sign of that charge. Also,  $\mathcal{N} \subset \mathbb{N} \cup \{0\}$  is the set of  $N$  indices, and we set  $\mathcal{N} \equiv \emptyset$  if  $N = 0$ , in which case  $\sum_{k \in \emptyset} (\dots) \equiv 0$ , so that the charge-free situation is included in (2). It is well-known, and readily verified, that (2) satisfies the *law of the conservation of electric charge*,  $d\mathbf{J} = \mathbf{0}$  in the sense of distributions, as demanded by the Ampère–Coulomb–Maxwell law.

By the manifestly covariant character of the Maxwell–Born–Infeld field laws with point sources, all Lorentz observers of any particular, actually realized electromagnetic structure in Minkowski spacetime satisfying these laws would necessarily conclude that they see their respective Lorentz frame manifestations of the *same* electromagnetic spacetime, whatever their relative states of uniform motion with respect to each other might be.

We end this subsection by recalling that  $\mathbf{F}$  is also exact and can therefore be written as the exterior derivative of a one-form, i.e.,  $\mathbf{F} = d\mathbf{A}$ , where  $\mathbf{A}$  is the *electromagnetic potential* on  $\mathbb{M}^4$ ; notice that while the exterior derivative is to be understood in the sense of distributions, away from the location of a charge the derivative exists in the regular sense, and since the singularities of  $\mathbf{F}$  are mild discontinuities, the one-form  $\mathbf{A}$  can be extended

continuously into the locations of the point charges. With the help of  $\mathbf{A}$  the law of motion for the point charges can be formulated.

### 2.1.2. The Law of Motion

The partial quantization of the classical theory is achieved by modifying the *law of motion* for the point charges. Like its classical counterpart, the quantum law of motion is formulated on  $\mathbb{M}_{\neq}^{4N}$ , the configuration space of  $N$  ordered world-points  $\mathbb{M}^{4N} = \times_{k=1}^N \mathbb{M}_k^4$  with all co-incidence points removed; here  $\mathbb{M}_k^4$  is the  $k$ th copy of  $\mathbb{M}^4$ . We first recall these basic ingredients of the classical law of motion. Its reformulation in terms of bundles on  $\mathbb{M}_{\neq}^{4N}$  will then naturally suggest a generalization which will reveal itself as a legitimate quantum law of motion for a single relativistic particle, and in the formal non-relativistic particle limit for many particles; modifications are necessary, however, for many relativistic particles.

#### *The Classical Law of Motion and Its Differential-Geometrical Reformulation.*

In ref. 29 we postulated fields  $\# \mathbf{A}$  on  $\mathbb{M}^4 \times \mathbb{M}_{\neq}^{4N}$  such that  $\# \mathbf{A}(\varpi, \varpi_1, \dots, \varpi_N)$  reduces to  $\mathbf{A}(\varpi)$  when the  $N$  configuration world-points  $\varpi_k \in H_k$ ,  $k = 1, \dots, N$ ; such fields can be defined only w.r.t. some space-like foliation of  $\mathbb{M}^4$ , by which we mean that  $\# \mathbf{A}(\varpi, \varpi_1, \dots, \varpi_N)$  reduces to  $\mathbf{A}(\varpi)$  when in addition to  $\varpi_k \in H_k$ ,  $k = 1, \dots, N$ , the  $N + 1$  world-points  $\{\varpi, \varpi_1, \dots, \varpi_N\}$  are picked from a constant-time  $t$  leaf of the smooth foliation; eventually we worked with the standard foliation. The subset in  $\mathbb{M}_{\neq}^{4N}$  of  $t$  leaf-synchronized configurations, which is diffeomorphic to  $\mathbb{R} \times \mathbb{R}^{3N}$ , will be called  $t$ -synchronized  $\mathbb{M}_{\neq}^{4N}$ , for brevity. Similarly we defined fields  $\# \mathbf{F} = \mathbf{d}\# \mathbf{A}$ , etc. This generic world configuration-indexed family of electromagnetic potentials  $\# \mathbf{A}$  defines  $N$  fields  $\tilde{\mathbf{A}}_k(\varpi_1, \dots, \varpi_N) \equiv \# \mathbf{A}(\varpi_k, \varpi_1, \dots, \varpi_N)$ ,  $k = 1, \dots, N$ . We further postulated that there is a scalar field  $\tilde{\Phi}$  on  $\mathbb{M}_{\neq}^{4N}$  such that  $\mathbf{d}_k \tilde{\Phi}(\varpi_1, \dots, \varpi_N) - z_k \alpha \tilde{\mathbf{A}}_k(\varpi_1, \dots, \varpi_N)$  is a Minkowski velocity co-vector field when the  $N$  world-points  $\varpi_1, \dots, \varpi_N$  are varied over a leaf of the foliation associated with the  $\#$ fields. It follows that  $\tilde{\Phi}$  has to obey the  $N$  equations

$$\star((\mathbf{d}_k \tilde{\Phi} - z_k \alpha \tilde{\mathbf{A}}_k) \wedge \star(\mathbf{d}_k \tilde{\Phi} - z_k \alpha \tilde{\mathbf{A}}_k)) = 1, \quad (3)$$

understood w.r.t. the foliation. Equations (3), which have a double root of which the future-oriented one is to be chosen, determine how the Minkowski-velocity co-vector fields evolve from one leaf of the foliation to another. The Minkowski-velocities co-vector field yields a first-order guiding law for the world-points  $\varpi_k = \eta_k(\tau)$  at proper-time  $\tau$ ; namely their Minkowski velocity co-vectors  $\mathbf{u}_k(\tau) = d\varpi_k/d\tau|_{H_k}$  obey

$$\mathbf{u}_k(\tau) = \mathbf{d}_k \tilde{\Phi}(\varpi_1, \dots, \varpi_N) - z_k \alpha \tilde{\mathbf{A}}_k(\varpi_1, \dots, \varpi_N), \quad (4)$$

where the  $\varpi_1, \dots, \varpi_N$  here are the world-points of the point charges on the same leaf.

While the  $\#$ fields and  $\tilde{\Phi}$  are defined w.r.t. a foliation, as explained at the beginning of the first subsection of 2.1.2, the actual electromagnetic output of the theory is Poincaré co-variant, as mentioned in the previous subsection. The theory is also manifestly Weyl-covariant, i.e., a gauge transformation

$$\#A(\varpi, \varpi_1, \dots, \varpi_N) \rightarrow \#A(\varpi, \varpi_1, \dots, \varpi_N) + dY(\varpi) \quad (5)$$

$$\tilde{\Phi}(\varpi_1, \dots, \varpi_N) \rightarrow \tilde{\Phi}(\varpi_1, \dots, \varpi_N) + \sum_k z_k \alpha Y(\varpi_k), \quad (6)$$

with any zero-form  $Y: \mathbb{M}^4 \rightarrow \mathbb{R}$  leaves  $\mathbf{F}$  and the  $\mathbf{u}_k$  invariant.

Next we recast (4) into a more geometrical format. Let  $\mathbf{U} = (\mathbf{u}_1, \dots, \mathbf{u}_N)$  consist of the  $N$  Minkowski-velocity co-vectors  $\mathbf{u}_k$  at the  $N$  world-points  $\varpi_k$  of the point-histories  $H_k$ ,  $k = 1, \dots, N$ , piercing a leaf. Furthermore, set  $\tilde{\mathbf{A}} = (\tilde{\mathbf{A}}_1, \dots, \tilde{\mathbf{A}}_N)$ , and let  $Z$  be a diagonal matrix, the  $k$ th  $4 \times 4$  block of which having entries  $z_k$  (in its diagonal), and let  $\mathbf{d}_{(N)}$  be Cartan's exterior derivative on  $\mathbb{M}^{4N}$ . With these definitions, it is now readily seen that the first order guiding laws (4) can be restated as

$$\mathbf{U} = e^{-i\tilde{\Phi}} (-i\mathbf{d}_{(N)} - \alpha Z \tilde{\mathbf{A}}) e^{i\tilde{\Phi}}, \quad (7)$$

i.e., the actual  $\mathbf{U}$  is set equal to the covariant logarithmic derivative, at the actual world-configuration on a leaf of a foliation, of a unitary section  $e^{i\tilde{\Phi}}$  of a complex line bundle on foliation-synchronized  $\mathbb{M}_{\neq}^{4N}$ , with  $(\alpha Z) \tilde{\mathbf{A}}$  the *electromagnetic connection* on the bundle.<sup>7</sup>

### The Quantum Law of Motion.

To begin with, we remark that the requirement that the section of the complex line bundle be unitary can be dropped without changing the content of the classical theory. Thus, instead of  $e^{i\tilde{\Phi}}$  we may consider a complex  $\tilde{\psi} \equiv \tilde{q}^{1/2} e^{i\tilde{\Phi}}$ , where  $\tilde{q}^{1/2}$  is a positive amplitude function on  $\mathbb{M}_{\neq}^{4N}$ , in terms of which (7) becomes

$$\mathbf{U} = \Re(\tilde{\psi}^{-1} (-i\mathbf{d}_{(N)} - \alpha Z \tilde{\mathbf{A}}) \tilde{\psi}), \quad (8)$$

where  $\Re$  means ‘‘real part.’’ Superficially (8) has a formal de Broglie–Bohm like appearance, but appearances are misleading, for so far  $\tilde{q}^{1/2}$  remains an undetermined, superfluous mathematical luxury which has no input into the theory whatever. Indeed, the condition that the  $N$  Minkowski co-vector

<sup>7</sup> The  $k$ th block component of the *electromagnetic curvature* of the bundle is (upto a factor  $z_k \alpha$ )  $\mathbf{d}_k \tilde{\mathbf{A}}_k = \tilde{\mathbf{F}}_k$ . Note that  $\tilde{\mathbf{F}}_k(\varpi_k) \neq \mathbf{F}(\varpi_k)$ , for the r.h.s. in this non-equation is ill-defined.

components of the right side of (8) give future-oriented time-like unit co-vector fields just gives back the Hamilton–Jacobi law for the phase  $\tilde{\Phi}$  of the bundle section, and (8) generates just the classical motions.

Next, we remark that we can re-calibrate the particles' time parameter  $\tau$  in (2) from being proper time into any other Lorentz-scalar time without changing the actual electromagnetic spacetime structure of the theory, whether it comes from a classical motion or not. To remain at the classical level of electromagnetic theory, the  $N$  Minkowski co-vector components of the right side of (8) then have to produce future-oriented time-like co-vector fields which are compatible with the new normalization of  $\mathbf{U}$  which is brought about by the re-calibration of  $\tau$  in the classical motions; of course, the  $\mathbf{u}_k$  are now in general no longer unit co-vectors. Moreover, the re-calibration will also entail a change of the foliation w.r.t. which the law of motion in configuration space is constructed. All this will change the appearance of the equations for  $\tilde{\Phi}$ , and the appearance of the law of motion, but it will not change the classical content of the theory.

Things will change, however, if we proceed with a re-calibration of  $\tau$  and the corresponding adjustments in the normalization of the guiding law on  $\mathbb{M}_{\neq}^{4N}$  without adjusting the foliation. In particular, since nothing prevents us *a priori* from linking the re-calibration of  $\tau$  to the amplitude field  $\tilde{Q}^{1/2}$  on  $\mathbb{M}_{\neq}^{4N}$ , now contemplate the following  $\tilde{Q}^{1/2}$ -dependent normalization of the future-oriented time-like co-vector fields defined by the r.h.s. of (8), namely for the  $k$ th component

$$\star((\mathbf{d}_k \tilde{\Phi} - z_k \alpha \tilde{\mathbf{A}}_k) \wedge \star(\mathbf{d}_k \tilde{\Phi} - z_k \alpha \tilde{\mathbf{A}}_k)) = 1 - \tilde{Q}^{-1/2} \square_k \tilde{Q}^{1/2}, \quad (9)$$

where  $\square_k$  is the wave operator for the  $k$ th world-point variable. The r.h.s. of (9) evaluated a-posteriori for the actual motions then yields the re-calibration of  $\tau$  along  $\mathbf{H}_k$  via  $\star(\mathbf{u}_k \wedge \star \mathbf{u}_k) = 1 - \tilde{Q}^{-1/2} \square_k \tilde{Q}^{1/2}|_{\mathbf{H}_k}$ . At this point, the amplitude field  $\tilde{Q}^{1/2} \geq 0$  may still be just arbitrarily prescribed. But now assume that instead of prescribing  $\tilde{Q}$ , we also postulate  $N$  individual “continuity equations” for  $\tilde{Q}$ , namely

$$\mathbf{d}_k(\tilde{Q} \star(\mathbf{d}_k \tilde{\Phi} - z_k \alpha \tilde{\mathbf{A}}_k)) = 0, \quad (10)$$

of course all of them understood w.r.t. a foliation. All these seem to be quite innocuous changes that just implement the freedom of having an amplitude factor  $\tilde{Q}^{1/2}$  and a re-calibration of  $\tau$  “at our disposal.” And yet, considered first for a single electron, and w.r.t. the standard foliation of spacetime, this actually achieves a quantization of the classical theory, albeit only a partial one, for spin and photon are not yet incorporated. Indeed, with respect to the standard foliation of spacetime, the section of



the complex line bundle now satisfies a Klein–Gordon equation, which is the “correct” relativistic wave equation for a spin-less point electron in interaction with electromagnetic fields. The quotes around “correct” are meant to remind the reader that there are no spin-less electrons in Nature, for which reason we do not really know what the correct wave equation would be if there were such a beast. Moreover, and the Pauli–Weisskopf work on the quantum field-theoretical interpretation of the Klein–Gordon equation aside, it is well-known that the quantum mechanical interpretation of the Klein–Gordon equation is burdened with the problem that this equation is of second order in time, not of first order. Be that as it may, it is certainly not an unreasonable first step to verify that the electromagnetic Maxwell–Born–Infeld field equations with point charge source can be consistently coupled with a de Broglie–Bohm like law of motion generated by a Klein–Gordon equation.

Unfortunately, the straightforward extension to the many electrons case, as formally depicted in (9) to (10), reveals problems with the synchronization that require the input of new ideas which lead to modifications of the simple scheme laid down above, and on which we briefly comment later on in the paper. Rewardingly though, if taken as heuristic starting point for a non-relativistic approximation to the equations of motion, formulas (9) to (10) are quite useful and lead to the many body Schrödinger equation coupled with the Maxwell–Born–Infeld #field equations.

### 3. THE ELECTROMAGNETIC QUANTUM THEORY ON A STANDARD FOLIATION

#### 3.1. The Actual Electromagnetic Field Laws

The field equations for the actual electromagnetic fields on spacetime  $\mathbb{M}^4$  are the Maxwell–Born–Infeld equations as listed in Section 4.2 of ref. 29. Very briefly, given point charges source terms

$$j(t, s) = \sum_{k \in \mathcal{A}} z_k \delta_{r_k(t)}(s), \quad (11)$$

$$\dot{j}(t, s) = \sum_{k \in \mathcal{A}} z_k \delta_{r_k(t)}(s) \dot{r}_k(t), \quad (12)$$

the evolution equation for  $D$ ,

$$\partial D = \nabla \times H - 4\pi j, \quad (13)$$

constrained by

$$\nabla \cdot \mathbf{D} = 4\pi j, \quad (14)$$

has to be solved together with the evolution equation for  $\mathbf{B}$ ,

$$\partial \mathbf{B} = -\nabla \times \mathbf{E}, \quad (15)$$

constrained by

$$\nabla \cdot \mathbf{B} = 0, \quad (16)$$

where the fields  $\mathbf{E}$  and  $\mathbf{H}$  are defined in terms of  $\mathbf{B}$  and  $\mathbf{D}$  by the aether laws

$$\mathbf{E} = \frac{\mathbf{D} - \beta^4 \mathbf{B} \times (\mathbf{B} \times \mathbf{D})}{\sqrt{1 + \beta^4 (|\mathbf{B}|^2 + |\mathbf{D}|^2) + \beta^8 |\mathbf{B} \times \mathbf{D}|^2}} \quad (17)$$

$$\mathbf{H} = \frac{\mathbf{B} - \beta^4 \mathbf{D} \times (\mathbf{D} \times \mathbf{B})}{\sqrt{1 + \beta^4 (|\mathbf{B}|^2 + |\mathbf{D}|^2) + \beta^8 |\mathbf{B} \times \mathbf{D}|^2}} \quad (18)$$

of Born and Infeld,<sup>(14)</sup> with  $\beta \in (0, \infty)$ .

Once  $\mathbf{B}$  and  $\mathbf{E}$  are known, one can also compute a magnetic vector potential  $A$  satisfying the evolution equation

$$\partial A = -\nabla A - \mathbf{E}, \quad (19)$$

and the constraint equation

$$\nabla \times A = \mathbf{B}, \quad (20)$$

where  $A$  satisfies the evolution equation of the Lorentz–Lorenz gauge,

$$\partial A = -\nabla \cdot A. \quad (21)$$

We recall that in the absence of any source terms, the resulting charge-free Maxwell–Born–Infeld field equations form a closed system of equations for the actual electromagnetic fields in  $\mathbb{M}^4$ . Also, the Maxwell–Born–Infeld field equations with *given* point charge sources are well-posed locally and can be solved “bottom-up,” at least in principle. Of course, the actual point charge sources in  $\mathbb{M}^4$  are only known after the motions of all point charges have been computed. As emphasized in the covariant section, this requires solving a whole family of Maxwell–Born–Infeld  $\#$ field equations with generic point sources. The solutions for the  $\#A$  and  $\#A$  then

define the solutions of (19) and (21) through conditioning with the actual motion, so that the actual Maxwell–Born–Infeld field equations get solved “top-down,” then.

### 3.2. The $t$ -Synchronized $\#$ Fields Equations

Also the  $\#$ fields equations are unaltered; cf. ref. 29. The  $t$ -synchronized space and time decomposition of  $\#A(\varpi, \varpi_1, \dots, \varpi_N)$  into components  $\#A$  and  $\#A$  gives

$$A^\#(t, s, \mathbf{S}) \equiv \#A(t, s, t_1, s_1, \dots, t_k, s_k, \dots, t_N, s_N) \Big|_{t_1=t_2=\dots=t_N=t}, \quad (22)$$

$$A^\#(t, s, \mathbf{S}) \equiv \#A(t, s, t_1, s_1, \dots, t_k, s_k, \dots, t_N, s_N) \Big|_{t_1=t_2=\dots=t_N=t} \quad (23)$$

on  $\mathbb{R} \times \mathbb{R}^{3(N+1)}$  (etc. for the other  $\#$ fields). As stipulated earlier, by conditioning with the actual configuration we want to obtain the actual fields on  $\mathbb{M}^4$  (in Lorentz gauge, say), i.e.,  $A^\#(t, s, \mathbf{R}(t)) = A(t, s)$  and  $A^\#(t, s, \mathbf{R}(t)) = A(t, s)$  (etc.). This canonically fixes the equations for the  $t$ -synchronized space and time decomposition of the  $\#$ fields. Namely,  $A^\#(t, s, \mathbf{S})$ ,  $A^\#(t, s, \mathbf{S})$ , and  $D^\#(t, s, \mathbf{S})$  satisfy the evolution equations

$$\partial A^\#(t, s, \mathbf{S}) = -V(t, \mathbf{S}) \cdot \nabla_S A^\#(t, s, \mathbf{S}) - \nabla \cdot A^\#(t, s, \mathbf{S}), \quad (24)$$

$$\partial A^\#(t, s, \mathbf{S}) = -V(t, \mathbf{S}) \cdot \nabla_S A^\#(t, s, \mathbf{S}) - \nabla A^\#(t, s, \mathbf{S}) - E^\#(t, s, \mathbf{S}), \quad (25)$$

$$\partial D^\#(t, s, \mathbf{S}) = -V(t, \mathbf{S}) \cdot \nabla_S D^\#(t, s, \mathbf{S}) + \nabla \times H^\#(t, s, \mathbf{S}) - 4\pi j^\#(t, s, \mathbf{S}), \quad (26)$$

where  $V(t, \mathbf{S}) \cdot \nabla_S \equiv \sum_{k=1}^N \mathbf{v}_k(t, \mathbf{S}) \cdot \nabla_k$  is the velocity field on configuration space that still needs to be defined; furthermore,  $D^\#(t, s, \mathbf{S})$  obeys the constraint equation

$$\nabla \cdot D^\#(t, s, \mathbf{S}) = 4\pi j^\#(t, s, \mathbf{S}), \quad (27)$$

where<sup>8</sup>

$$j^\#(t, s, \mathbf{S}) = \sum_{k \in \mathcal{A}} z_k \delta_{s_k}(s), \quad (28)$$

$$j^\#(t, s, \mathbf{S}) = \sum_{k \in \mathcal{A}} z_k \delta_{s_k}(s) \mathbf{v}_k(t, \mathbf{S}). \quad (29)$$

<sup>8</sup> Recall that the  $\#$ field re-formulation of the continuity equation of the charge conservation (in spacetime),  $\partial j^\#(t, s, \mathbf{S}) = -V(t, \mathbf{S}) \cdot \nabla_S j^\#(t, s, \mathbf{S}) - \nabla \cdot j^\#(t, s, \mathbf{S})$ , is an identity, not an independent equation.

The fields  $E^*(t, s, \mathbf{S})$  and  $H^*(t, s, \mathbf{S})$  in (25), (26) are defined in terms of  $D^*(t, s, \mathbf{S})$  and  $B^*(t, s, \mathbf{S})$  in precisely the same manner as the actual fields  $E(t, s)$  and  $H(t, s)$  are defined in terms of  $D(t, s)$  and  $B(t, s)$  through the Born–Infeld aether laws (17), (18), while  $B^*(t, s, \mathbf{S})$  in turn is defined in terms of  $A^*(t, s, \mathbf{S})$  in precisely the same manner as the actual  $B(t, s)$  is defined in terms of the actual  $A(t, s)$  in (20).

It is straightforward to verify that by substituting the actual configuration  $\mathbf{R}(t)$  for the generic  $\mathbf{S}$  in the  $t$ -synchronized  $\#$ fields satisfying the above equations, we obtain the actual electromagnetic potentials, fields, and charge-current densities satisfying the Maxwell–Born–Infeld field equations (in Lorentz–Lorenz gauge).

### 3.3. The Klein–Gordon Wave Function Formalism

As in the classical theory, conditioning  $A^*(t, s, \mathbf{S})$  and  $A^\#(t, s, \mathbf{S})$  with  $s = s_k$  for each  $k = 1, \dots, N$  gives the  $t$ -synchronized  $\tilde{A}_k$  and  $\tilde{A}_k^\#$  (etc.) fields on  $\mathbb{R} \times \mathbb{R}_\#^{3N}$ ,

$$\tilde{A}_k(t_1, s_1, \dots, t_N, s_N)|_{t_1=t_2=\dots=t_N=t} \equiv A_k(t, \mathbf{S}) \quad (30)$$

$$\tilde{A}_k^\#(t_1, s_1, \dots, t_N, s_N)|_{t_1=t_2=\dots=t_N=t} \equiv A_k^\#(t, \mathbf{S}). \quad (31)$$

So far everything that has been stated in standard space and time decomposition is exactly the same as in the classical theory. The new material starts next, revealing that the Hamilton–Jacobi equation has been replaced by a Klein–Gordon equation. We first consider the single electron theory, then the generalization to many electrons.

#### 3.3.1. A Single Electron

If there is only a single (positive or negative) electron in the world, no synchronization is necessary for  $\tilde{\Phi}$ , i.e.,  $\tilde{\Phi}(t, s_1) \equiv \Phi(t, s_1)$ . Then, in standard space and time decomposition, (9) becomes

$$-(-\partial\Phi - (\pm)\alpha A_1)^2 + |\nabla_1\Phi - (\pm)\alpha A_1|^2 + 1 - \varrho^{-1/2} \square_1 \varrho^{1/2} = 0, \quad (32)$$

with  $\square_1 = -\partial^2 + \nabla_1^2$ , and (10) becomes

$$\partial(\varrho(-\partial\Phi - (\pm)\alpha A_1)) + \nabla_1 \cdot (\varrho(\nabla_1\Phi - (\pm)\alpha A_1)) = 0. \quad (33)$$

Everywhere in the interior of  $\text{supp}(\varrho)$ , we now multiply (32) by  $\varrho^{1/2}e^{i\Phi}$ , and (33) by  $i\varrho^{-1/2}e^{i\Phi}$ , then add the so multiplied equations. The result is a single complex linear partial differential equation for

$$\varrho^{1/2}e^{i\Phi} \equiv \psi, \quad (34)$$

known as the Klein–Gordon equation

$$-(i\partial - (\pm)\alpha A_1)^2 \psi + (-i\nabla_1 - (\pm)\alpha A_1)^2 \psi + \psi = 0, \quad (35)$$

which is coupled self-consistently to the total electromagnetic \*potentials, and here restricted to the interior of  $\text{supp}(\varrho)$ . Conversely, inserting (34) into (35) and sorting into real and imaginary parts gives back the pair of equations (32) and (33). Note that  $|\psi|^2 = \varrho$  is a relativistic scalar, hence *not* a probability density, as could have seemed by the resemblance with Born's statistical law that  $|\Psi|^2(t, s_1)$  is a probability density for  $s_1$  when  $\Psi(t, s_1)$  is the Schrödinger wave function.

We next define the formal probability density field  $\varrho(-\partial\Phi - (\pm)\alpha A_1) \equiv \rho$  and the probability current vector-density field  $\varrho(\nabla_1\Phi - (\pm)\alpha A_1) \equiv \mathbf{j}^{\text{qu}}$  on configuration space, in terms of which (33) takes the familiar appearance of a continuity equation,

$$\partial\rho + \nabla_1 \cdot \mathbf{j}^{\text{qu}} = 0. \quad (36)$$

But having identified the evolution equations for  $\varrho$  and  $\Phi$  with the Klein–Gordon equation for  $\psi$ , we can express  $\rho(t, s_1)$  and  $\mathbf{j}^{\text{qu}}(t, s_1)$  directly in terms of  $\psi(t, s_1)$ , i.e.,

$$\rho = \Im(\bar{\psi}(-\partial - (\pm)\alpha A_1)\psi), \quad (37)$$

$$\mathbf{j}^{\text{qu}} = \Im(\bar{\psi}(\nabla_1 - (\pm)\alpha A_1)\psi), \quad (38)$$

where  $\Im$  means imaginary part, and we used the mathematical convention  $\bar{\psi}$  for the complex conjugate of  $\psi$  to avoid confusion with the star symbol for Hodge duals. The r.h.s. of (37) and the r.h.s. of (38) are recognized as the familiar expressions for the probability density and probability current density associated with the Klein–Gordon equation; notions that make sense as long as  $\rho \geq 0$ .

The ratio  $\mathbf{j}^{\text{qu}}(t, s_1)/\rho(t, s_1)$  defines the electron's quantum velocity field  $\mathbf{v}_1^{\text{qu}}(t, s_1)$ ,

$$\mathbf{v}_1^{\text{qu}} \equiv \frac{\Im(\bar{\psi}(\nabla_1 - (\pm)\alpha A_1)\psi)}{\Im(\bar{\psi}(-\partial - (\pm)\alpha A_1)\psi)}. \quad (39)$$

We remark that the  $\bar{\psi}$  in (39) can be replaced with  $\psi^{-1}$  simultaneously in numerator and denominator of the r.h.s. of (39), which is readily checked. We next note that rewritten in terms of  $\rho$  and  $\mathbf{v}_1^{\text{qu}}$ , (36) becomes

$$\partial\rho + \nabla_1 \cdot (\rho\mathbf{v}_1^{\text{qu}}) = 0, \quad (40)$$

and (40) implies that the initial  $\rho_0$  is being transported by the velocity field  $\mathbf{v}_1^{\text{qu}}$  so that it would seem to follow that  $\rho$  stays non-negative if it was so initially. In principle, however, “pathologies” of  $\mathbf{v}_1^{\text{qu}}$  may develop, *a priori* speaking.

In particular, the identification of  $s_1 \mapsto \mathbf{v}_1^{\text{qu}}(t, s_1)$  with a *velocity* field raises the question whether  $\mathbf{v}_1^{\text{qu}}(t, \cdot)$  remains subluminal if it is so initially, which is the case iff the Minkowski co-vector  $\mathfrak{I}(\bar{\psi}(\mathbf{d}_1 - (\pm) i\alpha\mathbf{A}_1) \psi)(t, \cdot)$  remains time-like at all  $s_1$ . Of course, if subluminality holds initially, one can argue that by continuity subluminality extends locally into the future, but whether this holds for all future times is not *a priori* clear. We note that even if eventually  $|\mathbf{v}_1^{\text{qu}}(t, \cdot)| > 1$  somewhere, the obvious next question is whether the actual point motion generated by  $\mathbf{v}_1^{\text{qu}}(t, \cdot)$  stays subluminal or whether it reaches those regions where  $\mathbf{v}_1^{\text{qu}}(t, \cdot)$  is superluminal.

### 3.3.2. Many Electrons

The many-electrons wave function formalism compounds the difficulties that the second time derivatives of the wave operators brings with it. Here we only comment very briefly on the many-electrons wave function formalism, leaving more detailed discussions for some future works.

To begin with, simply repeating the steps of the one-electron formalism now for the  $N$  particles situation, one concludes that in space and time decomposition the formal many-times functions  $\tilde{\Phi}(\dots, t_k, s_k, \dots)$  and  $\tilde{\mathcal{Q}}(\dots, t_k, s_k, \dots)$  combine into the formal many-times wave function

$$\tilde{\psi} \equiv \tilde{\mathcal{Q}}^{\frac{1}{2}} e^{i\tilde{\Phi}}, \quad (41)$$

which has to satisfy  $N$  many-times Klein–Gordon equations

$$-(i\partial_k - z_k \alpha A_k)^2 \tilde{\psi} + (-i\nabla_k - z_k \alpha A_k)^2 \tilde{\psi} + \tilde{\psi} = 0. \quad (42)$$

As in the classical theory, without further restriction the  $N$  Eq. (42) would overdetermine  $\tilde{\psi}$ , and so one should instead consider a wave function  $\psi(t, \mathbf{S}) = \tilde{\psi}(t, s_1, \dots, t, s_k, \dots)$  restricted to  $t$ -synchronized  $\mathbb{M}_{\neq}^{4N}$  ( $\cong \mathbb{R} \times \mathbb{R}_{\neq}^{3N} \subset \mathbb{M}_{\neq}^{4N}$ ). However, since (42) is of second order in time, it is less obvious now than in the single-electron case what the evolution equation for  $\psi$  should be; thus,  $\partial^2 \psi(t, \mathbf{S})$  involves mixed time derivatives of  $\tilde{\psi}(t, s_1, \dots, t, s_k, \dots)$ , which are not determined by (42). In the classical setting this problem does not arise because the equations for the classical  $\tilde{\mathcal{Q}}$  are of first order in time. Moreover, the additional problem arises that the obvious many-particles analogue of the velocity formula (39) that comes to mind, namely  $N$  “quantum velocities” fields

$$\mathbf{v}_k^{\text{qu}}(t, \mathbf{S}) = \frac{\mathfrak{I}(\tilde{\psi}(\nabla_k - iz_k \alpha A_k) \tilde{\psi})}{\mathfrak{I}(\tilde{\psi}(-\partial_k - iz_k \alpha A_k) \tilde{\psi})}, \quad (43)$$

is not acceptable, for it does not lead to a continuity equation for any reasonable choice of probability density  $\rho$  on configuration space  $\mathbb{R}_{\neq}^{3N}$ . The following three options are possible ways out of the dilemma.

First, in the context of test particle theory with given external fields, a many-electron Klein–Gordon formalism has been worked out<sup>(43)</sup> which operates with a single  $\rho$  and  $N$  currents  $\mathbf{j}_k^{\text{qu}}$  defined in terms of higher derivatives of  $\tilde{\psi}$ , and this formalism ought to be adaptable to our situation with total fields instead of externally given fields. Second, since the troubles come from having second-order time derivatives in the Klein–Gordon equation, one might think of using the familiar “square-root Klein–Gordon” equations instead. However, since already at the one-electron level such problems led Dirac to the invention of his first-order Dirac equation, one may want to take the above mentioned difficulties as an incentive to wait no longer but to now incorporate spin into the formalism, especially since this is the direction one will pursue eventually anyhow. Third, and last, one may want to take the formal relativistic many Klein–Gordon equations formalism as a heuristic starting point for a non-relativistic approximation, which essentially consists of (i) replacing the square of the first-order co-variant time derivative in (42) by twice the first-order co-variant time derivative itself, (ii) Born’s law for  $\rho$ , i.e.,  $\rho \equiv |\tilde{\psi}|^2$ , through which the r.h.s. of the formal velocity field (43) gets replaced by  $\mathfrak{I}(\tilde{\psi}^{-1}(\nabla_k - iz_k \alpha_k) \tilde{\psi})$ . Synchronization on  $\mathbb{R} \times \mathbb{R}_{\neq}^{3N}$  now does not run into any problems, and one obtains a many-body Schrödinger equation with potentials determined by the Maxwell–Born–Infeld  $^*\text{field}$  equations, the sources of which move according to the de Broglie–Bohm velocity field. Moreover, even though spin is not implemented then, the Pauli principle for fermions<sup>(41)</sup> can of course be implemented now, but also vindicated, as discussed in ref. 22. While we will work out the non-relativistic approximation explicitly only in the one-electron setting, the formal adaptation of this to the  $N$  electrons setting is then indeed straightforward.

### 3.4. The Cauchy Problem

The Cauchy problem of the charge-free situation is identical to the one of the classical theory, and need not be repeated here. Also when point charges are present, part of the Cauchy problem is still the same, too, namely the Cauchy problem for the  $^*\text{fields}$ ; however, the part of the Cauchy problem dealing with the fields on configuration space has changed radically. We only address the single electron version.

#### 3.4.1. The Configuration Space Problem

While in the classical single electron theory the Cauchy problem on configuration space dealt with a single field,  $\Phi$ , now we have two fields,

$\Phi$  and  $\varrho$ , or which is the same, one complex field  $\psi$ . This in itself is not a truly dramatic change, for also in the classical theory we could have added some luxury and amended  $\Phi$  by a *passive* scalar amplitude field  $\varrho$  satisfying (10) on the chosen foliation, which on the standard foliation ( $t$ -synchronization) becomes (33). The radical change thus comes about not from having another field  $\varrho$  and (33), the radical change comes about through the *second-order* term  $\varrho^{-1/2} \square_1 \varrho^{1/2}$  in (32). Indeed, in the amended-amplitude classical Hamilton–Jacobi theory the fact that the Hamilton–Jacobi equation for  $\Phi$  is of first order in time allows one to eliminate the time derivatives of  $\Phi$  in (33), as a consequence of which (33) in the classical amended single electron theory is a first-order equation for  $\varrho$ , given that  $\Phi$  satisfies the Hamilton–Jacobi PDE. In sharp contrast, in our single electron quantum theory the Cauchy problem is of second order for both  $\Phi$  and  $\varrho$ ; the highest time derivative of  $\varrho$  occurs now in (32), the highest time derivative of  $\Phi$  occurs now in (33). This turns the import of the two equations upside down.

We note that Eq. (39) allows us to couple the configuration space indexed family of Maxwell–Born–Infeld  $\#$ field equations with point sources directly to the Klein–Gordon equation, without having any recourse whatsoever to  $\varrho$  and  $\Phi$ . We remark that any solution  $\psi$  which does not develop a zero in the interior of its support then maps into a unique global solution pair<sup>9</sup>  $\varrho, \Phi$  of (33) and (32), as verified by retracing backward the steps that led us to the Klein–Gordon equation.

The Klein–Gordon equation (35), as a second order equation in time, requires initial data  $\psi(0, \cdot)$  and  $\partial\psi(0, \cdot)$ . The fields  $A_1$  and  $A_1$  that enter (35) are obtained by conditioning from the respective  $\#$ fields with  $s = s_1$ ; the  $\#$ fields in turn satisfy the first order equations (24)–(29), with  $v_1^{\text{qu}}(t, s_1)$  given in (39). Initial data for all evolution equations have to be given. It is straightforward to check that the initial value problem for the Maxwell–Born–Infeld  $\#$ field equations with point sources moving according to the velocity field (39) and with the wave function  $\psi$  satisfying the Klein–Gordon equation is well-defined. Whether the Cauchy problem leads to global or just local existence and uniqueness results is an interesting open problem.

The Cauchy problem described is autonomous in the sense that the actual electromagnetic spacetime does not figure. Of course, to obtain the actual electromagnetic spacetime from a solution of the  $\#$ fields–Klein–Gordon equations, data for the  $\#$ fields have to reduce to the data for the

<sup>9</sup> We are not aware of general results as to which initial conditions for the Klein–Gordon equation lead to zeros of  $\psi$  and which do not when the potentials  $A_1$  and  $A_1$  are given, not to speak of the self-consistent situation in which the potentials have to be solved for simultaneously with  $\psi$ .



actual fields when the actual particle configuration is substituted for the generic one, and data for  $\psi$  and  $\partial\psi$  need to give the actual initial velocity of the point charge.

### 3.4.2. The Actual Motion and the Actual Fields

Once  $v_1^{\text{qu}}(t, s_1)$  has been computed autonomously by solving the coupled system of Klein–Gordon and  $\#$ field equations, one finally can solve the relativistic de Broglie–Bohm type guiding equation with given initial data  $r_1(0)$  to obtain the actual trajectory  $t \mapsto r_1(t)$  of the point charge, i.e.,  $r_1(t)$  satisfies

$$\dot{r}_1(t) = v_1^{\text{qu}}(t, r_1(t)). \quad (44)$$

Once this has been done, the actual point sources in  $\mathbb{M}^4$  are known, too, and given by  $j(t, s) = (\pm) \delta_{r_1(t)}(s)$  and  $\mathbf{j}(t, s) = (\pm) \delta_{r_1(t)}(s) \dot{r}_1(t)$ , with  $r_1(t)$  satisfying (44). These are indeed the familiar expressions (11) and (12) for the electric “density” and electric current “vector density” of a single point charge at  $s_1 = r_1(t)$  moving with velocity  $\dot{r}_1(t)$ . It is straightforward to verify that charge conservation is guaranteed.

This fully vindicates our designation of  $v_1^{\text{qu}}(t, r_1(t))$  as the velocity of the electron in the quantum theory.

Having the actual point-charge source terms for the actual Maxwell–Born–Infeld field equations, we could now solve them bottom-up to get the actual fields; however, as already emphasized, by the very setup of the theory we can simply substitute the actual position  $r_1(t)$  for the generic  $s_1$  in the  $\#$ fields to obtain the actual Maxwell–Born–Infeld fields. Thus,  $A(t, s) = \#A(t, s, r_1(t))$ , etc., which solves the actual Maxwell–Born–Infeld field equations top-down.

## 4. APPLICATION TO ATOMS

While the absence of spin in the Klein–Gordon equation and the absence of photons from the electromagnetic fields limit the applicability of our theory in practical situations, we do get the correct low-energy physics whenever spin effects and the photonic nature of the electromagnetic fields are known to contribute only small corrections. Thus, in the non-relativistic limit we obtain the correct Schrödinger equation with Coulomb interaction. Since in our theory the electromagnetic fields are the total fields, and the self-field energies are all finite, the Coulomb interactions emerge in the non-relativistic limit *without* any truncation and renormalization. We shall work this out explicitly for the hydrogen atom, a non-genuinely electromagnetic example for which it can be assumed that the

nucleus and the electron move at non-relativistic speeds. To keep the presentation as simple as possible, we actually treat the nucleus in the Born–Oppenheimer approximation as infinitely massive; we also assume the nucleus to be a point without magnetic moment. The extensions of all the genuinely electromagnetic formulas to this non-genuine setting are straightforward. The many-electrons atom with nuclear charge  $z > 1$  and  $N = z$  negative electrons will be treated elsewhere.

To leading order in an expansion in terms of powers of  $\alpha$ , assumed to be small, the familiar data of non-relativistic quantum theory emerge in the formal limit  $\beta \downarrow 0$  if and only if we identify  $\alpha$  with Sommerfeld’s fine structure constant—as we have argued non-rigorously already in ref. 29. A non-vanishing Born’s aether constant  $\beta$  in turn induces corrections to the spectrum, which must be small, and this puts some rough and ready upper bounds on  $\beta$ .

#### 4.1. The Hydrogen Atom

For the hydrogen atom, in Born–Oppenheimer approximation, the infinitely massive point nucleus of charge  $+1$  can be assumed to be at rest at the origin  $\theta$  of our space. This requires adding a term  $\delta_o(s)$  to the charge density. The single point electron moves along the trajectory  $t \mapsto s_1 = r_1(t)$  (henceforth, we drop the suffix  $_1$  from  $r_1(t)$ ), and the charge density and current vector density then read

$$j(t, s) = \delta_o(s) - \delta_{r(t)}(s) \quad (45)$$

$$\mathbf{j}(t, s) = -\delta_{r(t)}(s) \dot{\mathbf{r}}(t). \quad (46)$$

These are now the point source terms for the Maxwell–Born–Infeld field equations, which are supplemented by the asymptotic conditions that all fields vanish at spatial infinity. The total electric and magnetic fields have potentials which in turn enter the Klein–Gordon equation. More precisely, what enters the Klein–Gordon equation are not the actual fields for the unknown actual position and velocity, but the conditioned  $\#$ fields for the generic positions on configuration space and the velocities associated to them by the velocity field  $\mathbf{v}^{\text{qu}}$ . Solutions of the Klein–Gordon equation (35) on single-electron configuration space define the velocity vector field (39) on that configuration space, which evolves any actual electron’s position vector  $\mathbf{r}(t)$  via the relativistic de Broglie–Bohm type guiding equation  $\dot{\mathbf{r}}(t) = \mathbf{v}^{\text{qu}}(t, \mathbf{r}(t))$ , which in turn determines, for each actual trajectory the electron traces out, the point source terms (45) and (46) for the Maxwell–Born–Infeld field equations, closing the loop. However, since the Maxwell–Born–Infeld  $\#$ field equations have to be solved along with the

Klein–Gordon equation, the actual Maxwell–Born–Infeld field equations need not to be solved again; their solution is simply obtained then by substituting the actual trajectory for the generic one in the respective solutions of the  $\#$ fields equations.

#### 4.1.1. Existence of Infinitely Many Radiation-Free Bound States

We first establish the existence of stationary solutions of our coupled system of equations. Stationarity in the Lorentz gauge with asymptotically (at spatial infinity) vanishing conditions for the  $\#$ potentials means that the electric potential  $\#A(t, s, s_1) \equiv \#A_0(s, s_1)$ , and the magnetic vector potential  $\#\mathbf{A}(t, s, s_1) \equiv \#\mathbf{A}_0(s, s_1)$ . The potential terms in the Klein–Gordon equation are then explicitly time-independent, and the only time-dependence allowed is in an overall phase rotation of the wave function, thus  $\psi(t, s_1) = e^{-iet}\psi^{\text{stat}}(s_1)$ . This in turn implies  $\mathbf{v}^{\text{qu}}(t, s_1) \equiv \mathbf{0}$ , which implies  $\#\mathbf{j}(t) = \mathbf{0}$  (wherever  $s_1$  may be), and this now implies  $\#\mathbf{j}(t, s, s_1) = \delta_o(s) - \delta_{s_1}(s)$ . Having static sources together with  $\partial\#\mathbf{D} \equiv \mathbf{0}$  implies  $\nabla \times \#\mathbf{H} \equiv \mathbf{0}$ , whence  $\#\mathbf{H} \equiv \mathbf{0}$ , whence  $\#\mathbf{B} \equiv \mathbf{0}$ , and therefore  $\#\mathbf{A} \equiv \mathbf{0}$ .

Hence, the only allowed fields are electrostatic, and we need to solve the electrostatic Coulomb–Born–Infeld equation

$$-\nabla \cdot \frac{\nabla \#A_0(s, s_1)}{\sqrt{1 - \beta^4 |\nabla \#A_0(s, s_1)|^2}} = 4\pi\delta_o(s) - 4\pi\delta_{s_1}(s) \quad (47)$$

for arbitrary location  $s_1$  of the electron, with asymptotic condition  $\#A_0(s, s_1) \rightarrow 0$  as  $|s| \rightarrow \infty$ . Any such solution of (47) is unique, see our proof in ref. 29. As for the existence of solutions, we invoke Bartnik’s remark in ref. 1 that his existence Theorem 5.4 generalizes to maximal space-like slices with light cone singularities to anticipate the existence of electrostatic potentials with two point charges for any configuration of the electron and nucleus positions; however, an explicit existence proof should be supplied eventually.

Of the solution  $s \mapsto \#A_0(s, s_1)$  to (47) only  $\#A_0(s_1, s_1) \equiv A_0(s_1)$  is needed in the Klein–Gordon equation.<sup>10</sup> Interestingly enough,  $A_0(s_1)$  can be calculated without knowledge of the complete solution  $s \mapsto \#A_0(s, s_1)$  of (47), as we show next.

For the purpose of calculating  $A_0(s_1)$ , we remark that the solution to (47) must satisfy

$$-\frac{\nabla \#A_0}{\sqrt{1 - \beta^4 |\nabla \#A_0|^2}} = \#\mathbf{D}_{\text{Coulomb}}^{(2)} + \nabla \times \#\mathbf{Z} \quad (48)$$

<sup>10</sup> Strictly speaking, to keep with our notational conventions, instead of  $A_0(s_1)$  we should write  $A_{1,0}(s_1)$ , but no confusion should arise from dropping the suffix 1 here.

for all  $s \neq 0$  or  $s_1$ , where

$$\#D_{\text{Coulomb}}^{(2)}(s, s_1) = -\nabla(|s|^{-1} - |s - s_1|^{-1}), \quad (49)$$

so that for a given solution  $\#A_0$  of (47),  $\nabla \times \#Z$  is uniquely defined by (48) for all  $s \neq 0$  or  $s_1$ ; and since  $\beta^2 |\nabla \#A_0(s, s_1)| \rightarrow 1$  when  $s \rightarrow 0$  or  $s_1$ , we can extend  $\nabla \times \#Z$  continuously to all  $s \in \mathbb{R}^3$  by setting  $\nabla \times \#Z(s, s_1) = 0$  for  $s = 0, s_1$ . The field  $s \mapsto \#Z(s, s_1)$  (with  $s_1$  as parameter) is itself an electrostatic vector potential which we can assume to vanish for  $|s| \rightarrow \infty$ . Note that  $\#Z$  is defined only up to the gauge transformation  $\#Z \rightarrow \#Z + \nabla \#U$ , under which (48) is invariant. We can remove this freedom by imposing the gauge condition  $\nabla \cdot \#Z = 0$ , which can always be achieved by solving a Poisson equation for  $\#U$ , if necessary. However, what matters is only  $\nabla \times \#Z$ . Easily inverting (48) we get

$$-\nabla \#A_0 = \frac{\#D_{\text{Coulomb}}^{(2)} + \nabla \times \#Z}{\sqrt{1 + \beta^4 |\#D_{\text{Coulomb}}^{(2)} + \nabla \times \#Z|^2}}, \quad (50)$$

and integration along any path from  $s_0$  to  $s$  gives the identity  $\#A_0(s, s_1) = \#A_0(s_0, s_1) + \int_{s_0}^s \nabla \#A_0(\hat{s}, s_1) \cdot d\hat{s}$ , with the integrand given by the r.h.s. of (50). Of course,  $s_0$  should be picked conveniently so that  $\#A_0(s_0, s_1) = 0$ ; for instance, the standard convention (valid for more than two point charges as well) would be to let  $s_0 \rightarrow \partial\mathbb{R}^3$  (infinity). However, notice that by the symmetry of the problem we know that  $\#A_0(s_0, s_1) = 0$  for all  $s_0 \in \{0.5s_1 + s^\perp, s^\perp \cdot s_1 = 0\}$ , and so we may want to pick such an  $s_0$ . Furthermore, a straightforward calculation shows that on the straight line joining the nuclear point charge (representing the proton) and the point electron, we have

$$\nabla \times \frac{\#D_{\text{Coulomb}}^{(2)}(s, s_1)}{\sqrt{1 + \beta^4 |\#D_{\text{Coulomb}}^{(2)}(s, s_1)|^2}} \Big|_{s \in \{\xi s_1; \xi \in \mathbb{R}\}} = 0, \quad (51)$$

so that we may in fact conclude that

$$\nabla \times \#Z(s, s_1) \Big|_{s \in \{\xi s_1; \xi \in \mathbb{R}\}} = 0. \quad (52)$$

Hence, picking  $s_0 = 0.5s_1$ , we find the explicit one-dimensional integral formula

$$A_0(s_1) = -\int_{1/2}^1 \frac{s_1 \cdot \#D_{\text{Coulomb}}^{(2)}(\xi s_1, s_1)}{\sqrt{1 + \beta^4 |\#D_{\text{Coulomb}}^{(2)}(\xi s_1, s_1)|^2}} d\xi, \quad (53)$$

as announced. (By the symmetry of the configuration, the integral can be replaced by  $1/2$  the same integral taken in the limits from 0 to 1.)

The integral (53) can be manipulated into a form that shows a strong resemblance to integrals listed in ref. 26 which can be evaluated in closed form with the help of elliptic integrals and elementary functions; however, we have not yet succeeded to evaluate it in closed form. Therefore we have to resort to computing  $A_0(s_1)$  for small and large values of  $|s_1|/\beta$ . To state the proposition, we recall that

$$A_{\text{Born}}^{(\pm)}(s) = (\pm) \frac{1}{\beta} \int_{|s|/\beta}^{\infty} \frac{dx}{\sqrt{1+x^4}} \quad (54)$$

is Born's solution for the electrostatic potential of a single positive or negative electron at the origin of space.

**Proposition 4.1.** If the electron is near the nucleus, more precisely iff  $|s_1| < 2\sqrt{2}\beta$ , then  $A_0(s_1)$  can be expanded into a convergent series in powers of  $|s_1|/\beta$ , i.e.,

$$A_0(s_1) = -\frac{1}{2\beta} \left[ \frac{|s_1|}{\beta} - \frac{|s_1|^5}{\beta^5} \int_{1/2}^1 \frac{\xi^4(1-\xi)^4}{(1-2\xi(1-\xi))^2} d\xi + O\left(\frac{|s_1|^9}{\beta^9}\right) \right]. \quad (55)$$

If, on the other hand, the electron is far from the nucleus, i.e., for  $|s_1| \geq 2\sqrt{2}\beta$ , then  $A_0(s_1)$  can be expanded (asymptotically exact for  $|s_1| \rightarrow \infty$ ) to get

$$A_0(s_1) = A_{\text{Born}}^{(-)}(\theta) + \frac{1}{|s_1|} \left[ 1 - U\left(\frac{\beta}{|s_1|}\right) \right], \quad (56)$$

with  $U(\beta/|s_1|) < 0$  and  $|U(\beta/|s_1|)| < 2\beta/|s_1|$  for  $|s_1|$  large enough.

**Remark 4.2.** We can take (56) as defining  $U(\beta/|s_1|)$  for all  $|s_1|$ , and indeed  $U(\beta/|s_1|)$  is then well defined for all  $|s_1| > 0$ . Also, using (55) one easily sees that then also  $U(\beta/|s_1|) = 1 + |s_1| A_{\text{Born}}^{(-)}(\theta) + \frac{1}{2} \frac{|s_1|^2}{\beta^2} + \dots$  for small  $|s_1|$ . It follows that  $U$  is bounded. It also follows that  $U(\beta/|s_1|) > 0$  for small  $|s_1|$ , and since  $U(\beta/|s_1|) < 0$  for large  $|s_1|$ , we believe that  $U(\beta/|s_1|) = 0$  for exactly one  $|s_1|$ ; however, so far we have not been able to prove this.

*Proof of Proposition 4.1.* The integral formula (53) can be easily rendered as

$$A_0(s_1) = -\frac{|s_1|}{\beta^2} \int_{1/2}^1 \left( 1 + \frac{|s_1|^4}{\beta^4} \frac{\xi^4(1-\xi)^4}{(1-2\xi(1-\xi))^2} \right)^{-1/2} d\xi. \quad (57)$$

The Taylor expansion of (57) for  $|s_1| < 2\sqrt{2}\beta$  is elementary and gives (55). Note that only powers  $(|s_1|/\beta)^{4k+1}$ ,  $k = 0, 1, 2, \dots$  enter, and that all coefficients are integrals of bounded rational functions which can be evaluated in closed form.

To obtain the asymptotic expansion of  $A_0(s_1)$  for  $|s_1| \rightarrow \infty$ , we rewrite (53) as

$$A_0(s_1) = \frac{1}{\beta} \int_{2\sqrt{2}\beta/|s_1|}^{\infty} \frac{f'(y)}{\sqrt{1+y^4}} dx, \quad (58)$$

where  $xy = \beta/|s_1|$ , and  $f'$  means the derivative of  $f$ , with

$$f(y) = \sqrt{\frac{1}{4} + y^2 - y\sqrt{1+y^2}}. \quad (59)$$

Writing out  $f'(y)$  explicitly is not very illuminating. Fortunately, all we need are the following features of the map  $y \mapsto f'(y)$ , which are straightforward to prove:

- (i)  $y \mapsto f'(y) + 1$  is strictly negative on  $(0, 1/2\sqrt{2})$ ;
- (ii)  $y \mapsto f'(y)$  is decreasing on  $[0, 1/2\sqrt{2})$  and strictly decreasing on  $(0, 1/2\sqrt{2})$ ;
- (iii)  $y \mapsto f'(y)$  is strictly concave on  $[0, 1/2\sqrt{2})$ ;
- (iv)  $y \mapsto f'(y)$  is analytic on  $(0, 1/2\sqrt{2})$ , and the Taylor–MacLaurin expansion about  $y = 0$  of  $f'(y)$  reads (with  $y \geq 0$ )

$$f'(y) = -1 - \frac{3}{2}y^2 - 4y^3 - \frac{75}{8}y^4 + O(y^5); \quad (60)$$

- (v)  $y \mapsto f'(y)$  is bounded above and below on  $[0, 1/2\sqrt{2})$  by

$$-\frac{1}{\sqrt{3}} \frac{1}{\sqrt{1-2\sqrt{2}y}} \geq f'(y) \geq -\frac{1}{\sqrt{3}} \frac{1 + \frac{3}{2}(1-2\sqrt{2}y)}{\sqrt{1-2\sqrt{2}y}}, \quad (61)$$

and the difference of left- and right-hand sides  $\downarrow 0$  as  $2\sqrt{2}y \uparrow 1$ .

We now prove first the leading order asymptotics in (56), i.e., we show that

$$\lim_{|s_1| \rightarrow \infty} A_0(s_1) = A_{\text{Born}}^{(-)}(\theta). \quad (62)$$

Indeed, by (i) we have  $f'(y) \leq -1$ . Inserting this estimate in (58) gives

$$A_0(s_1) \leq -\frac{1}{\beta} \int_{2\sqrt{2}\beta/|s_1|}^{\infty} \frac{1}{\sqrt{1+x^4}} dx, \quad (63)$$

and taking the limit gives

$$\limsup_{|s_1| \rightarrow \infty} A_0(s_1) \leq A_{\text{Born}}^{(-)}(\theta). \quad (64)$$

On the other hand, for any small  $\epsilon > 0$ , if  $x \geq \epsilon$  then  $y \leq \beta/(|s_1|\epsilon)$ ; hence, for fixed  $\epsilon > 0$  and any  $x \geq \epsilon$  we have  $\lim_{|s_1|/\beta \rightarrow \infty} f'(y) = -1$ , uniformly, and this gives us

$$\lim_{|s_1| \rightarrow \infty} \frac{1}{\beta} \int_{\epsilon+2\sqrt{2}\beta/|s_1|}^{\infty} \frac{f'(y)}{\sqrt{1+x^4}} dx = -\frac{1}{\beta} \int_{\epsilon}^{\infty} \frac{1}{\sqrt{1+x^4}} dx \geq A_{\text{Born}}^{(-)}(\theta), \quad (65)$$

while

$$\begin{aligned} & \liminf_{|s_1| \rightarrow \infty} \frac{1}{\beta} \int_{2\sqrt{2}\beta/|s_1|}^{\epsilon+2\sqrt{2}\beta/|s_1|} \frac{f'(y)}{\sqrt{1+x^4}} dx \\ & \geq \liminf_{|s_1| \rightarrow \infty} -\frac{5}{2\sqrt{3}} \frac{1}{\beta} \int_{2\sqrt{2}\beta/|s_1|}^{\epsilon+2\sqrt{2}\beta/|s_1|} \frac{\sqrt{x}}{\sqrt{x-2\sqrt{2}\beta/|s_1|}} dx \\ & \geq \liminf_{|s_1| \rightarrow \infty} -\frac{5}{\sqrt{3}} \frac{1}{\beta} \sqrt{\epsilon} \sqrt{\epsilon+2\sqrt{2}\frac{\beta}{|s_1|}} = -\frac{5}{\sqrt{3}} \frac{1}{\beta} \epsilon. \end{aligned} \quad (66)$$

for all  $\epsilon$ . In (66), we used the estimate  $\sqrt{1+x^4} \geq 1$  in conjunction with (i), followed by the lower estimate for  $f'(y)$  in (v) (which we further estimated by discarding the positive  $y$  contribution in the numerator), followed by an elementary integration by parts and the omission of a manifestly positive additive integral. Estimates (64) on the one hand, and (65) and (66) on the other, establish (62).

To establish the next-to-leading order term, we now show that

$$\lim_{|s_1| \rightarrow \infty} |s_1| (A_0(s_1) - A_{\text{Born}}^{(-)}(\theta)) = 1. \quad (67)$$

We first notice that  $|s_1| (A_0(s_1) - A_{\text{Born}}^{(-)}(\theta))$  depends on  $\beta$  and  $|s_1|$  exclusively through the combination  $\beta/|s_1|$ , and since  $\beta/|s_1| \downarrow 0$  as  $|s_1| \uparrow \infty$ , it is convenient here to introduce the abbreviation  $\beta/|s_1| \equiv \zeta$ . The limit  $|s_1| \uparrow \infty$  then becomes the limit  $\zeta \downarrow 0$ . To carry out this limit we split the integral for  $A_{\text{Born}}^{(-)}(\theta)$  in two,

$$A_{\text{Born}}^{(-)}(\theta) = -\frac{1}{\beta} \int_0^{2\sqrt{2}\beta/|s_1|} \frac{1}{\sqrt{1+x^4}} dx - \frac{1}{\beta} \int_{2\sqrt{2}\beta/|s_1|}^{\infty} \frac{1}{\sqrt{1+x^4}} dx, \quad (68)$$

then lump the (negative of the) second integral together with the integral (58) for  $A_0(s_1)$ , while the (negative of the) first integral will be handled on its own. After multiplication of these integrals by  $|s_1|$ , we change integration variables  $x \rightarrow x/\zeta \equiv \xi$  in the (negative of the) first integral of (68) and  $x \rightarrow y (= 1/\xi)$  in the ‘‘lumped’’ integral. After this reshuffling of terms, the l.h.s. in (67) becomes a sum of two limits  $\zeta \rightarrow 0$  which can be carried out easily using monotone convergence,

$$\begin{aligned} \lim_{\zeta \rightarrow 0} \left( \int_0^{2\sqrt{2}} \frac{d\xi}{\sqrt{1+\zeta^4 \xi^4}} + \int_0^{1/2\sqrt{2}} \frac{1+f'(y)}{\sqrt{1+\zeta^4/y^4}} \frac{dy}{y^2} \right) \\ = \int_0^{2\sqrt{2}} d\xi + \int_0^{1/2\sqrt{2}} \frac{1+f'(y)}{y^2} dy. \end{aligned} \quad (69)$$

The first of these limiting integrals obviously equals  $2\sqrt{2}$ . As for the second limiting integral, note that by the Taylor series of  $f'(y)$  about  $y=0$  its integrand is regular at  $y=0$ , while the bounds in (v) show that the singularity at  $y \uparrow 1/2\sqrt{2}$  is a reciprocal square root, hence the integrand is integrable there, too. By (i) this integral is negative. To evaluate this integral, we write it as a limit of an integral as the lower limit of integration of that integral tends to null, which gives (after carrying out one obvious integration)

$$\int_0^{1/2\sqrt{2}} \frac{1+f'(y)}{y^2} dy = -2\sqrt{2} + \lim_{\epsilon \downarrow 0} \left( \frac{1}{\epsilon} + \int_{\epsilon}^{1/2\sqrt{2}} \frac{f'(y)}{y^2} dy \right). \quad (70)$$

To evaluate the remaining integral in (70), we go through a sequence of successive changes of integration variable, first  $y \rightarrow \tilde{y} = y + \sqrt{1+y^2}$ , next  $\tilde{y} \rightarrow \hat{y} = \tilde{y}^2 - 1$ , finally  $\hat{y} \rightarrow u = \hat{y}^{-2} - 1$ , ending up with the elementary integral



$$\begin{aligned} \int_{\epsilon}^{1/2\sqrt{2}} \frac{f'(y)}{y^2} dy &= - \int_0^{[2\epsilon(\epsilon+\sqrt{1+\epsilon^2})]^{-2}-1} \frac{du}{\sqrt{u}} \\ &= -\frac{1}{\epsilon} \sqrt{(\epsilon+\sqrt{1+\epsilon^2})^{-2}-4\epsilon^2} = -\frac{1}{\epsilon} + 1 + O(\epsilon). \end{aligned} \quad (71)$$

With (71), the limit  $\epsilon \downarrow 0$  in (70) gives us

$$\int_0^{1/2\sqrt{2}} \frac{1+f'(y)}{y^2} dy = 1 - 2\sqrt{2}, \quad (72)$$

which together with (69) proves (67).

As for the term  $U(\beta/|s_1|) = 1 - |s_1| (A_0(s_1) - A_{\text{Born}}^{(-)}(\theta))$  in (56), we write it as the difference of the r.h.s. (69) and the expression between big parentheses on the l.h.s. of (69),

$$U(\zeta) = \int_0^{2\sqrt{2}} \left(1 - \frac{1}{\sqrt{1+\zeta^4\xi^4}}\right) d\xi + \int_0^{1/2\sqrt{2}} \left(1 - \frac{1}{\sqrt{1+\zeta^4/y^4}}\right) \frac{1+f'(y)}{y^2} dy, \quad (73)$$

then use property (i) for  $f'$ , then the convexity estimate  $1 - \sqrt{1+\zeta^4\xi^4}^{-1} \leq \frac{1}{2}\zeta^4\xi^4$  and drop a negative term to estimate

$$\begin{aligned} |U(\zeta)| &\leq \int_0^{2\sqrt{2}} \left(1 - \frac{1}{\sqrt{1+\zeta^4\xi^4}}\right) d\xi + \int_0^{1/2\sqrt{2}} \left(1 - \frac{1}{\sqrt{1+\zeta^4/y^4}}\right) \left|\frac{1+f'(y)}{y^2}\right| dy \\ &\leq \frac{\zeta^4}{2} \left(\int_0^{2\sqrt{2}} \xi^4 d\xi - \int_{\zeta}^{1/2\sqrt{2}} \frac{1+f'(y)}{y^6} dy\right) - \int_0^{\zeta} \frac{1+f'(y)}{y^2} dy. \end{aligned} \quad (74)$$

The remaining integrals in (74) containing  $f'$  evaluate to

$$\begin{aligned} &\frac{\zeta^4}{2} \int_{\zeta}^{1/2\sqrt{2}} \frac{1+f'(y)}{y^6} dy - \int_0^{\zeta} \frac{1+f'(y)}{y^2} dy \\ &= \frac{3}{2}\zeta + 2\zeta^2 + O(\zeta^3) + \frac{1}{4}\zeta + \zeta^2 + O(\zeta^3) \end{aligned} \quad (75)$$

where we used property (iv) and the fact that by formula (59) for  $f$  singularities occur only at  $y = \pm i$  and  $y = 1/2\sqrt{2}$ . Finally,

$$\int_0^{2\sqrt{2}} \xi^4 d\xi = \frac{1}{5} (2\sqrt{2})^5. \quad (76)$$

Adding up the numbers and rounding up, and recalling that  $\zeta = \beta/|s_1|$  and that  $1 + f' < 0$ , we find

$$-2 \frac{\beta}{|s_1|} + O\left(\frac{\beta^2}{|s_1|^2}\right) < U\left(\frac{\beta}{s_1}\right) < 20 \frac{\beta^4}{|s_1|^4}. \quad (77)$$

Finally, we sharpen the bound from below on  $U$  by noting that for  $\zeta = \beta/|s_1| \ll 1$ , we have  $1 - \sqrt{1 + \zeta^4 \zeta^4}^{-1} = \frac{1}{2} \zeta^4 \zeta^4 + O(\zeta^8)$ , while the integral containing  $1 + f'$  in (73) is  $O(\zeta)$ ; hence

$$U(\zeta) = -\zeta \frac{3}{2} \int_0^\infty \left(1 - \frac{t^2}{\sqrt{1+t^4}}\right) dt + O(\zeta^2), \quad (78)$$

Hence,  $U(\beta/|s_1|) < 0$  for  $|s_1|$  big enough.

This concludes the proof of our Proposition 4.1.  $\blacksquare$

We believe that Proposition 4.1 yields the first rigorous and explicit results about the electrostatic two-body problem for the nonlinear Maxwell–Born–Infeld field equations. More important is the fact that the asymptotic expansion (56) of  $A_0(s_1)$  for  $|s_1| \rightarrow \infty$  given in Proposition 4.1 verifies that the theory produces the correct Coulomb law for the dependence on  $|s_1|$  of the electrostatic configurational energy of two point charges which are far apart. We remark that if we would have obtained a different asymptotic power law than  $1/|s_1|$  for  $A_0(s_1) - A_{\text{Born}}^{(-)}(\theta)$ , or the correct power law but with a coefficient different from unity, the theory would not be able to reproduce the known physical data correctly. While the asymptotic expansion (56) of  $A_0(s_1)$  is therefore a gratifying result to have, upon reflection it is actually somewhat surprising that it is true at all!—for it means that as  $|s_1| \rightarrow \infty$ , the nonlinearity of the Maxwell–Born–Infeld field equations shows in the leading, but not in the next-to-leading order term of the asymptotic expansion, only to show again in the next-to-next-to-leading order. This seems like a curious behavior for a nonlinear field theory.

The next step is to show that for the two-body solution  ${}^{\#}A_0(s, s_1)$  of (47), the stationary Klein–Gordon equation

$$-A_1 \psi^{\text{stat}}(s_1) + (1 - |\varepsilon + \alpha A_0(s_1)|^2) \psi^{\text{stat}}(s_1) = 0 \quad (79)$$

with asymptotic condition  $\psi^{\text{stat}}(s_1) \rightarrow 0$  for  $|s_1| \rightarrow \infty$  admits bound states. As the most immediate spin-off of Proposition 4.1 we indeed have

**Corollary 4.3.** The Klein–Gordon equation (79) has infinitely many bound states.

*Proof of Corollary 4.3.* Rewriting the stationary Klein–Gordon equation (79) as

$$-\frac{1}{2} \Delta_1 \psi^{\text{stat}}(s_1) - (\bar{\varepsilon} \alpha \bar{A}_0(s_1) + \frac{1}{2} \alpha^2 \bar{A}_0(s_1)^2) \psi^{\text{stat}}(s_1) = \frac{1}{2} (-1 + \bar{\varepsilon}^2) \psi^{\text{stat}}(s_1), \quad (80)$$

with  $\bar{A}_0(s_1) = A_0(s_1) - A_{\text{Born}}^{(-)}(\mathbf{0})$  and  $\bar{\varepsilon} = \varepsilon + \alpha A_{\text{Born}}^{(-)}(\mathbf{0})$ , we see that on the left-hand side of (80) we have a Schrödinger operator with Schrödinger potential  $V_{\bar{\varepsilon}}(s_1) = -\bar{\varepsilon} \alpha \bar{A}_0(s_1) - \frac{1}{2} \alpha^2 \bar{A}_0(s_1)^2$ . Hence we can interpret (80) as a Schrödinger eigenvalue problem with a one-parameter family of potentials  $V_{\bar{\varepsilon}}$  together with the constraint that the eigenvalues  $E$  have to be of the form  $E = \frac{1}{2} (-1 + \bar{\varepsilon}^2)$ . Now, by Proposition 4.1,  $V_{\bar{\varepsilon}}$  is the sum of a Rollnik potential and an arbitrarily small bounded potential. Furthermore, again by Proposition 4.1, we have  $V_{\bar{\varepsilon}}(s_1) \leq -C |s_1|^{-2+\delta}$  for  $|s_1| > R_0$  and all  $\bar{\varepsilon} > 0$ . Hence, by Theorem XIII.6a of ref. 36 we conclude that for any  $\bar{\varepsilon} > 0$  the Schrödinger operator on the l.h.s. of (80) has infinitely many negative eigenvalues  $E_k(\bar{\varepsilon})$ ,  $k \in \mathbb{N}$ . Furthermore, since  $-\frac{1}{2} \Delta + V_{\bar{\varepsilon}}$  in (80) is a compact perturbation of  $-\frac{1}{2} \Delta$ , its infinite discrete spectrum accumulates at zero. Moreover, by a simple monotonicity argument we have  $E_k(\bar{\varepsilon}) \downarrow$  as  $\bar{\varepsilon} \uparrow$ , while  $\frac{1}{2} (-1 + \bar{\varepsilon}^2) \uparrow$  as  $\bar{\varepsilon} \uparrow$ ; in particular  $\frac{1}{2} (-1 + \bar{\varepsilon}^2)$  ranges from  $-\frac{1}{2}$  to 0 as  $\bar{\varepsilon}$  moves from  $0^+$  to 1. ■

Two remarks: First, one should be able to prove that the lowest eigenvalue of (79) in fact corresponds to the radiation-free ground state of hydrogen. For this purpose one has to discuss the manifestly positive energy functional which we defer to a separate publication. Second, the normalization of the stationary  $\psi^{\text{stat}}$  follows from Eq. (37); however, for the spectral calculations we may set  $\|\psi^{\text{stat}}\|_{L^2} = 1$ .

#### 4.1.2. The Discrete Spectrum (Estimating $\beta$ )

Proposition 4.1 says that the value of the total electrostatic potential at the position of the point electron varies with the distance  $|s_1|$  from the nucleus according to Coulomb's law when the electron is "far away from the nucleus." The error bound on the asymptotic expansion (56) tells us more specifically how far away from the nucleus the electron needs to be so that Coulomb's law holds within any given relative error. This error bound together with the impressive range of validity of Coulomb's law will translate into some upper bound for  $\beta$  through comparison with the well-known spectral data for hydrogen. The assumption of an infinitely massive nucleus does not invalidate these arguments.

Now, we do have a fully special-relativistic theory, but we have not incorporated spin. It would thus be foolish to aim at a comparison with the relativistic details of the quantum mechanical hydrogen spectrum as computed from Dirac's equation with purely Coulombic potential<sup>(23, 28)</sup> (not to speak of the fine details caused by the Lamb shift<sup>(23)</sup>). Yet, it seems reasonable to demand that the  $\beta$ -induced corrections to the hydrogen spectrum as given to leading order in  $\alpha$ , i.e.,  $O(\alpha^2)$ , by Schrödinger's equation with purely Coulombic potential, should be sub-dominant to the known  $O(\alpha^4)$  corrections computed from Dirac's equation with purely Coulombic potential.

To expand (80) to leading order in  $\alpha$  we make the Ansatz  $\psi^{\text{stat}} = \psi_n + \delta\psi^{\text{stat}}$ , with  $|\delta\psi^{\text{stat}}| \ll 1$ , and  $\bar{\varepsilon} = 1 + E_n$ , with  $|E_n| \ll 1$ , and discard all terms that are subordinate to the respective leading order terms (this also means to discard the  $\alpha^2 \bar{A}_0^2(s_1)$  term in the Schrödinger potential). This leads to

$$-\frac{1}{2} A_1 \psi_n(s_1) - \alpha \bar{A}_0(s_1) \psi_n(s_1) = E_n \psi_n(s_1). \quad (81)$$

Here,  $n$  refers to the  $n$ th bound state, of which there are infinitely many by repeating the arguments of Corollary 4.3; we do not display the degeneracy of the spectrum. Next, to estimate  $\beta$ , notice that only  $\bar{A}_0(s_1)$  enters in (81), and since by Proposition 4.1 and its ensuing remark we have  $\bar{A}_0(s_1) = |s_1|^{-1} (1 - U(\beta/|s_1|))$ , with  $U(\beta/|s_1|)$  bounded and satisfying  $|U(\beta/|s_1|)| \leq 2\beta/|s_1|$  for small  $\beta/|s_1|$ , we may now take the limit  $\beta \downarrow 0$  in which  $\bar{A}_0(s_1) \rightarrow |s_1|^{-1}$ , obtaining the Schrödinger equation for the Coulomb Hamiltonian

$$-\frac{1}{2} \Delta_1 \psi_n^{(0)}(s_1) - \alpha |s_1|^{-1} \psi_n^{(0)}(s_1) = E_n^{(0)} \psi_n^{(0)}(s_1) \quad (82)$$

with

$$E_n^{(0)} = -\frac{\alpha^2}{2(n+1)^2}, \quad n = 0, 1, 2, \dots \quad (83)$$

(The indexing is chosen to meet the convention that  $E_0$  means the ground state energy.) We pause for a moment to remark that our calculation so far rigorously vindicates the identification of  $\alpha$  with Sommerfeld's fine structure constant.

Next, applying standard first-order perturbation theory,<sup>(36)</sup> we can write  $\psi_n(s_1) = \psi_n^{(0)}(s_1) + \delta\psi_n^{(\beta)}(s_1)$  and  $E_n = E_n^{(0)} + \delta E_n^{(\beta)}$  to compute the  $\beta$ -induced correction  $\delta E_n^{(\beta)}$  to  $E_n^{(0)}$  to first order in  $U$  as

$$\delta E_n^{(\beta)} = \alpha \int_{\mathbb{R}^3} |\psi_n^{(0)}(s_1)|^2 |s_1|^{-1} U(\beta/|s_1|) d^3(s_1). \quad (84)$$

We only estimate the ground state energy correction, i.e., (84) with  $n = 0$ , for which

$$\psi_0^{(0)}(s_1) = \sqrt{\alpha^3/\pi} e^{-\alpha|s_1|} \quad (85)$$

is the familiar normalized eigenfunctions of the ground state for (82). Proposition 4.1 and the remark thereafter (also note that (73) yields  $|U| \leq 2\sqrt{2}$ ) now give

$$\begin{aligned} |\delta E_0^{(\beta)}| &\leq \alpha 2\sqrt{2} \int_{\{|s_1| < R\}} |\psi_0^{(0)}(s_1)|^2 |s_1|^{-1} d^3(s_1) \\ &\quad + 2K\alpha\beta \int_{\{|s_1| \geq R\}} |\psi_0^{(0)}(s_1)|^2 |s_1|^{-2} d^3(s_1), \end{aligned} \quad (86)$$

for  $R$  and  $K \geq 1$  big enough, and  $R$  is to be chosen so that this expression is minimized. For this purpose, we will assume that  $\beta$  is not bigger than 1, vindicate this assumption a posteriori, then bootstrap to a smaller estimate for  $\beta$ . Thus, assuming  $\beta \leq 1$ , we find  $R = \beta K/\sqrt{2}$  (to leading order), which leads to the moderately accurate estimate

$$|\delta E_0^{(\beta)}| \lesssim 4K\alpha^3\beta, \quad (87)$$

which, when  $\beta \leq C\alpha$ , is indeed of  $O(\alpha^2)$  relative to the leading Coulomb term, which is  $\propto \alpha^2$ . (Actually, (87) should be an upper bound for  $\delta E_0^{(\beta)}$ .) To bootstrap the estimate for  $\beta$  further, note that the  $\beta$ -induced correction (87) should definitely not compete with the first relativistic Dirac–Coulomb correction to the non-relativistic Schrödinger–Coulomb spectrum, which enters as  $\delta E_0^{DC} = \frac{1}{4}\alpha^4$ . Hence, with  $K = 1$  for concreteness, we need to have at least  $\beta \leq 10^{-1}\alpha$ ; equality would result in roughly equal contributions by both corrections. However, since the Dirac correction gives excellent agreement, we can actually be confident that such a conservative estimate for the  $\beta$ -induced correction is too feeble and can be improved significantly. If a  $\beta$ -induced correction of not more than about one-hundredth of the first Dirac correction is allowed, (87) would give  $\beta \leq 10^{-3}\alpha$ , roughly. Less aggressively,  $\beta \leq C\alpha$ , with  $C$  less than 10, seems reasonable. Born's value for  $\beta$  is compatible with these estimates, but so is the temptingly speculative thought that, perhaps,  $\beta = \alpha$ . In any event, it is also conceivable that better estimates with a complete model with spin will give  $\beta \ll \alpha$ , which would rule out Born's value of  $\beta$ , and at the same time put an end to the speculation whether  $\beta = \alpha$ .

We finally translate these estimates into actual distances beyond which Coulomb's law is valid. By Proposition 4.1, we have that  $A_0(s_1) - A_{\text{Born}}^{(-)}(0) = |s_1|^{-1}$  within 1% relative error when  $|s_1| \geq 200\beta$  in leading order

asymptotics; cf. with the validity of the Taylor expansion of  $A_0(s_1)$  for  $|s_1| \leq 2\sqrt{2}\beta \approx 2.8\beta$ . Recall that the Compton wave length of the electron is our unit of length; relative to this unit, the so-called classical electron radius equals Sommerfeld's fine structure constant  $\alpha$ , while the Bohr radius equals  $1/\alpha$ . A value of  $\beta \approx \alpha$  would mean that Coulomb's law for the pair potential between point nucleus and point electron is valid to within 1% relative error down to a distance of about 200 classical electron radii.

#### 4.1.3. The Hydrogen Atom in Small-Velocities Approximation

Different from the stationary state treatment, where the velocities of all charges were identically zero, to obtain the formulas for the regime of non-zero but small velocities we here resort to non-rigorous but very plausible arguments; again, while non-rigorous, we see no reason why one should not be able to make these arguments rigorous.

Thus, we not only assume that radiation-reaction can be neglected; indeed to get the leading order effects we may even *assume* that magnetic effects can be neglected. That is, we assume that the electron moves so slowly that the electromagnetic potential for the total electromagnetic fields equals, at the position of the electron, the electrostatic Maxwell–Born–Infeld potential for the instantaneous configuration. For the remaining steps we adapt Dirac's prescriptions for the Dirac equation as reproduced in conventional textbooks.<sup>11</sup> Thus, to obtain the non-relativistic limit of the Klein–Gordon equation (35) on single-electron configuration space with an infinitely massive nucleus at the origin, we insert the Ansatz

$$\psi(t, s_1) = e^{-i\frac{5}{2}t} \check{\psi}(t, s_1) \quad (88)$$

into (35). In (88),  $\check{\psi}(t, s_1)$  is slowly varying in time as compared to  $e^{-i\frac{5}{2}t}$ , having an even slower varying time derivative; the unusual looking  $5/2$  is due to a term  $\alpha A_{\text{Born}}^{(-)}(\theta) = -3/2$  (assuming  $\beta_{\text{Born}}$ ). In (35), after factoring out  $e^{-i\frac{5}{2}t}$  the terms of the type const.  $\check{\psi}$  then cancel out, and we are left with

$$(-\partial^2 + 2(1 + \alpha\bar{A}_0(s_1))i\partial + 2\alpha\bar{A}_0(s_1) + \alpha^2\bar{A}_0^2(s_1) + A_1) \check{\psi}(t, s_1) = 0. \quad (89)$$

Next, invoking the so-called singular perturbation theory we may neglect the second-order time derivative as small versus the first-order time derivatives; we also neglect the  $O(\alpha)$  term versus the  $O(1)$  term in the coefficient of the first-order time derivative, which is justified as long as the electron stays sufficiently far away from the nucleus—with our estimates on  $\beta$  this means farther than a few electron Compton wavelengths, which are

<sup>11</sup> We alert the reader to the fact that some prescriptions for the Klein–Gordon equation (with given fields) that one can find in some otherwise excellent textbooks are not correct.

distances a factor 10 to  $10^2$  smaller than the Bohr radius of the hydrogen atom; and we neglect the  $O(\alpha^2)$  term versus the  $O(\alpha)$  term in the Schrödinger potential. Then  $\check{\psi} \approx \Psi$ , with  $\Psi$  solving Schrödinger's equation for the hydrogen atom with Coulomb–Born–Infeld potential,

$$i\partial_t \Psi(t, s_1) = -\frac{1}{2} \Delta \Psi(t, s_1) - \alpha \bar{A}_0(s_1) \Psi(t, s_1). \quad (90)$$

Recalling our large distance asymptotics  $\bar{A}_0(s_1) \sim |s_1|^{-1}$ , which becomes exact for all  $s_1$  in the limit  $\beta \downarrow 0$  and which holds very accurately in the range of validity of (90), we see that at the same level of accuracy we may now replace (90) by Schrödinger's equation with the traditional Coulomb potential  $-\alpha |s_1|^{-1}$  as Schrödinger potential.

Finally, in the same non-relativistic approximation, the relativistic guiding equation for the (negative) electron, (44) with  $v^{\text{qu}}$  given by (39), reduces to the de Broglie–Bohm guiding equation

$$\dot{r}(t) = \mathfrak{I}(\Psi^{-1} \nabla_1 \Psi)(t, r(t)). \quad (91)$$

## 5. ON THE DE BROGLIE–BOHM GUIDING LAW

The Schrödinger equation (90) (with the further approximation  $\bar{A}_0(s_1) \approx |s_1|^{-1}$ ) is of course well accepted as a basic equation of the non-relativistic quantum mechanics of the hydrogen atom (with infinitely massive nucleus). The guiding equation (91) is not. It was first proposed by de Broglie<sup>(16)</sup> (see also Chaps. 6, 9, and 10 in ref. 17) who, however, did not pursue this lead any further until his idea was re-discovered and its merits explained in great detail by Bohm,<sup>(6)</sup> upon which de Broglie himself returned to this approach.<sup>(18)</sup>

At the non-relativistic level, and generalized to the many-body situation, the corresponding Schrödinger equation together with the corresponding many-body de Broglie–Bohm guiding equation on configuration space, have been shown to provide an unorthodox yet entirely consistent and paradox-free formulation of non-relativistic spin-less quantum mechanics in terms of which all the usual measurement axioms of the conventional formulation can be explained as effective rules of procedure<sup>(2, 6, 20, 24)</sup>—to the extent that can reasonably be expected from a non-relativistic theory. It has been extended to include spin via the many-body Pauli equation and a generalization of (91) involving the Pauli spinors.

Yet, Einstein for instance considered it “too cheap” a trick to get rid of the “measurement problem,”<sup>12</sup> and while one can only speculate about

<sup>12</sup> The collection of Bell's articles<sup>(2)</sup> is mandatory reading. A very good collection of publications by Bell and almost everyone else about the measurement problem is ref. 45.

Einstein's reasons for his assessment, many physicists have raised a similarly spirited objection, that the de Broglie–Bohm guiding equation for the point particle motions seemed to be “artificially appended” to the autonomous Schrödinger dynamics, apparently without feedback from the contemplated particle motions. To the extent that a feedback from the actual motion of the one and only real configuration of the particles in the world is meant, such a feedback does not exist, indeed. Yet a hint of some feedback loop from generic motions of proper point particles into the Schrödinger equation could logically have been seen in the expression for the Coulomb energy of proper point charge configurations that enters the Schrödinger equation for atoms and molecules. In any event, as long as a consistent dynamical theory of point charges and electromagnetic fields was not available, this feedback loop remained speculative. Although explicitly shown here only for the hydrogen atom, the de Broglie–Bohm formulation of non-relativistic quantum mechanics with Coulomb Hamiltonian and autonomous Schrödinger dynamics obtains in the non-relativistic limit of our least invasively quantized relativistic electromagnetic field theory with point charges. While in our formulation the actually real particles and fields configuration of the world does not enter the system of Maxwell–Born–Infeld  $\#$ field plus Klein–Gordon equations,<sup>13</sup> in this system no dynamical equation is truly autonomous in itself, with  $\psi$  providing the guiding field for the generic point charge sources of the  $\#$ field equations, the solutions of which in turn providing the potentials within the Klein–Gordon equation. In this sense a certain amount of feedback from the guiding field over the  $\#$ fields back into Schrödinger's equation exists. Our present work thereby lends new support to the de-Broglie–Bohm formulation of non-relativistic quantum mechanics: since in the relativistic theory the set of dynamical variables comes as a closed package from which no subset of it may be left out, its non-relativistic limit now shines a new light on the de-Broglie–Bohm guiding equation in which it no longer appears as artificially appended to the Schrödinger equation.

## 6. SUMMARY AND OUTLOOK

In this paper, we succeeded in the partial quantization of the UV problem-free classical theory of electromagnetism with point charges that

<sup>13</sup> The initial conditions for the actual electromagnetic fields are inherited to some extent by the initial conditions for the  $\#$ fields through the condition that these initial  $\#$ fields become the actual initial fields when the actual initial particle configuration is substituted for the generic one.



we developed in ref. 29. Here, “partial” refers to the fact that only the degrees of freedom of point charge motion are affected by the quantization procedure. Our procedure is “least invasive” in the sense that it does not tamper with the integrity of the classical mathematical structures which guarantee the absence of divergence problems for the classical theory. We emphasize that at no point in our procedure have we replaced the mathematical objects of the classical theory by operators. This guarantees that the mathematical integrity of the whole formalism is left intact, which is what we mean by least invasive quantization. Singularities feature merely as mild defects in the electromagnetic potentials, and since no UV infinities are associated with them, no regularization of the defects and renormalization of parameters is called for.

The theory developed in ref. 29 and in this paper produces a relativistically covariant actual electromagnetic spacetime, but the laws of motion require a foliation which has to be granted a certain reality of its own; cf. also refs. 4, 21, and 25. The formalism should apply to the physics of positive or negative point electrons to the extent that spin effects and the photonic nature of the electromagnetic fields can be neglected; however, the relativistic quantum theory is worked out explicitly here only for a single electron coupled to its fields.

In both the classical and the quantum theory, the actual electromagnetic fields are solutions of the Maxwell–Born–Infeld field equations with the actual point charges as sources, which move according to relativistic guiding laws. In the classical theory, the guiding law is of Hamilton–Jacobi type, generated by a solution of a relativistic Hamilton–Jacobi PDE coupled self-consistently to a configuration space indexed family of electromagnetic potentials satisfying corresponding field equations. In the quantum theory, the guiding law for a single point charge reveals itself as of relativistic de Broglie–Bohm type, with the guiding field generated by a solution of a relativistic Klein–Gordon PDE coupled to the configuration indexed field equations. The relativistic many electrons wave function formalism requires modifications that we briefly commented on; however, in the non-relativistic particles limit the many electrons formalism can be worked out along the lines of this paper. In that regime only minor modifications of the theory are needed to accommodate the electromagnetic effects of other, non-genuinely electromagnetic particles, such as nuclei with or without magnetic moment and form factor, spinning or not. This requires putting in by hand the parameters  $z_k$  for the charge number,  $\kappa_k$  for the ratio of the the electron’s to the  $k$ th nucleus’ rest mass, and, if desired, a smeared-out spinning charge distribution.

We remark that the dynamical equations of the quantum theory presented here are as deterministic as those of the classical theory, with a well-defined joint Cauchy problem for the wave function, the electromagnetic fields, and the point charges. In the non-relativistic limit these equations reduce to the dynamical equations of the de-Broglie–Bohm formulation of non-relativistic quantum mechanics, for which its Cauchy problem with Coulomb and Newton interactions has been proved to be globally well-posed.<sup>(3)</sup> It is therefore reasonable to expect that at least local well-posedness of our relativistic Cauchy problem can be proved to hold; global well-posedness is not necessarily to be expected in a relativistic theory.

Another rewarding aspect of our formalism is that the quantum theory does not require postulating any additional “measurement axioms;” more poetically speaking, it does not suffer from “the malaise of the measurement problem.”<sup>(46)</sup> The dynamical equations of the theory themselves say what happens in a measurement, in principle at least. Whether the theory does make all the same predictions that the conventional relativistic quantum formalism without spin and photon would make is to be doubted, at least we do not see any reason why literally the same predictions should come out. In any event, while our theory certainly achieves a certain consistent generalization of the de-Broglie–Bohm formulation of non-relativistic quantum mechanics with Coulomb interactions to the relativistic purely electromagnetic world, without the approximation of involving only given fields, it is a difficult open problem of exactly how much of the non-relativistic Bohmian quantum formalism can be consistently reconciled with relativity theory.<sup>(4, 21, 25)</sup>

As a by-product of finally having a dynamically consistent formulation of electrodynamics with point charge(s), we are now also able for the first time to address the problem of the correct value of Born’s aether constant  $\beta$ , which enters the theory through the Born–Infeld laws of the aether. We found in ref. 29 that Born’s computation, based as it is on his dynamically incomplete formulation of the theory, is inconclusive. Born calculated the value of  $\beta$  by identifying the empirical electron rest mass  $m_e$  ( $\times c^2$ ) with the electrostatic energy of his spherically symmetric electrostatic solution for a single point charge. Our calculations here are based on the non-relativistic hydrogen spectrum, for which we worked out the first explicit and rigorous results in the static two-body problem for the Maxwell–Born–Infeld field equations. While our estimates for  $\beta$  leave the value calculated by Born viable for now, the definitive calculation of  $\beta$  can be done only after spin, and perhaps also the photon, have been incorporated into the theory. These are the issues that we will take up next.

## EPILOGUE

“I will say, though, that my feeling is that one of the really unsolved problems is to include the EM field. You have also banged your head against it. ... This is not a criticism of the Bohm formalism. It is a statement that the problem is known (to some of us) to be serious and has to be addressed and it is not easy at all.”

Elliott H. Lieb (private communication, August 29, 2001)

By implementing the notion of the point electron consistently into the classical relativistic theory of electromagnetism and then, through least invasive quantization, into a spin- and photon-less quantum theory, we also accomplished a consistent implementation of the total relativistic electromagnetic fields into the Bohm formalism of quantum theory. Yet these are only the first steps. The incorporation of spin and the photon have to be addressed, and other burning issues, such as pair creation and annihilation, will hopefully be understood along these lines as well, in due course. Meanwhile I hope that Elliott H. Lieb, to whom this two-parts paper is dedicated in admiration on occasion of his 70th birthday, will take some pleasure in the fact that his comments and advice, offered at a very early stage of this work, have played a not unimportant rôle in its creation. Elliott has been a constant inspiration and encouragement, and I hope that he will continue to inspire and encourage us all for many many more years to come.

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## REFERENCES

1. R. Bartnik, Existence of maximal surfaces in asymptotically flat spacetimes, *Commun. Math. Phys.* **94**:155–175 (1984).
2. J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics* (Cambridge University Press, Cambridge, UK, 1987).
3. K. Berndl, D. Dürr, S. Goldstein, G. Peruzzi, and N. Zanghì, On the global existence of Bohmian mechanics, *Commun. Math. Phys.* **173**:647–673 (1995).
4. K. Berndl, D. Dürr, S. Goldstein, and N. Zanghì, Nonlocality, Lorentz invariance, and Bohmian quantum theory, *Phys. Rev. A* **53**:2062–2073 (1996).
5. I. Białyński-Birula, Nonlinear electrodynamics: Variations on a theme by Born and Infeld, in *Quantum Theory of Particles and Fields*, Special Volume in Honor of Jan Łopuszański, B. Jancewicz and J. Lukierski, eds. (World Scientific, Singapore, 1983), pp. 31–48.
6. D. Bohm, A tentative interpretation of the quantum theory in terms of hidden variables. Part I, *Phys. Rev.* **85**:166–179 (1952); Part II, *ibid.* **85**:180–193 (1952).
7. G. Boillat, Nonlinear electrodynamics: Lagrangians and equations of motion, *J. Math. Phys.* **11**:941–951 (1970).
8. M. Born, Zur Quantenmechanik der Stossvorgänge, *Z. Phys.* **37**:863–867 (1926a); Quantenmechanik der Stossvorgänge, *Z. Phys.* **38**:803–827 (1926b).
9. M. Born, Modified field equations with a finite radius of the electron, *Nature* **132**:282 (1933).
10. M. Born, On the quantum theory of the electromagnetic field, *Proc. Roy. Soc. London Ser. A* **143**:410–437 (1934).
11. M. Born, Théorie non-linéaire du champ électromagnétique, *Ann. Inst. H. Poincaré* **7**:155–265 (1937).
12. M. Born, *Atomic Physics*, 8th rev. ed. (Blackie & Son, Glasgow, 1969).
13. M. Born and L. Infeld, Electromagnetic mass, *Nature* **132**:970 (1933).
14. M. Born and L. Infeld, Foundation of the new field theory, *Nature* **132**:1004 (1933); *Proc. Roy. Soc. London Ser. A* **144**:425–451 (1934).
15. M. Born and L. Infeld, On the quantization of the new field equations. Part I, *Proc. Roy. Soc. London Ser. A* **147**:522–546 (1934); Part II, *Proc. Roy. Soc. London Ser. A* **150**:141–166 (1935).
16. L. V. de Broglie, La structure de la matière et du rayonnement et la mécanique ondulatoire, *Comptes Rendus* **184**:273–274 (1927); La mécanique ondulatoire et la structure atomique de la matière et du rayonnement, *J. Phys. et Rad.* **8**:225–241 (1927); see also *Reports on the 1927 Solvay Congress* (Gauthier-Villars, Paris, 1927).
17. L. V. de Broglie, *An Introduction to the Study of Wave Mechanics* (E. P. Dutton and Co., New York, 1930).
18. L. V. de Broglie, *La physique quantique restera-t-elle indéterministique?* (Gauthier-Villars, Paris, 1953).
19. P. A. M. Dirac, A reformulation of the Born–Infeld electrodynamics, *Proc. Roy. Soc. London Ser. A* **257**:32–43 (1960).
20. D. Dürr, S. Goldstein, and N. Zanghì, Quantum equilibrium and the origin of absolute uncertainty, *J. Stat. Phys.* **67**:843–907 (1992).
21. D. Dürr, S. Goldstein, K. Münch-Berndl, and N. Zanghì, Hypersurface Bohm–Dirac models, *Phys. Rev. A* **60**:2729–2736 (1999).
22. D. Dürr, S. Goldstein, J. Taylor, and N. Zanghì, *Bosons, Fermions, and the Natural Configuration Space of Identical Particles*, preprint (Rutgers University, March 2003).
23. J. Glimm and A. Jaffe, *Quantum Physics*, 2nd ed. (Springer, New York, 1980).

24. S. Goldstein, Quantum theory without observers, Part I, *Phys. Today* (March 1998); Part II, *ibid.* (April 1998).
25. S. Goldstein and R. Tumulka, Opposite arrows of time can reconcile relativity and nonlocality, *Class. Quant. Grav.* **20**:557–564 (2003).
26. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 4th. ed. (Academic Press, San Diego, 1980).
27. J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975); 3rd ed. (Wiley, New York, 1999).
28. S. Keppeler, Semiclassical quantisation rules for the Dirac and Pauli equations, *Ann. Phys. (N.Y.)* **304**:40–71 (2003).
29. M. K.-H. Kiessling, Electromagnetic field theory without divergence problems. 1. The Born legacy, *J. Stat. Phys.* **116**:1057–1122 (2004).
30. C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, New York, 1973).
31. J. Plebański, *Lecture Notes on Nonlinear Electrodynamics* (NORDITA, Copenhagen, 1970) (quoted in ref. 5).
32. M. H. L. Pryce, On a uniqueness theorem, *Proc. Cambridge Phil. Soc.* **31**:625–628 (1935).
33. M. H. L. Pryce, Commuting coordinates in the new field theory, *Proc. Roy. Soc. London Ser. A* **150**:166–172 (1935).
34. M. H. L. Pryce, On the new field theory, *Proc. Roy. Soc. London Ser. A* **155**:597–613 (1936).
35. M. H. L. Pryce, On the new field theory. II. Quantum theory of field and charges, *Proc. Roy. Soc. London Ser. A* **159**:355–382 (1937).
36. M. Reed and B. Simon, Analysis of operators, in *Methods of Modern Mathematical Physics IV* (Academic Press, Orlando, 1978).
37. E. Schrödinger, Contribution to Born's new theory of the electromagnetic field, *Proc. Roy. Soc. London Ser. A* **150**:465–477 (1935).
38. E. Schrödinger, Non-linear optics, *Proc. Roy. Irish Acad. Sect. A* **47**:77–117 (1942).
39. E. Schrödinger, Dynamics and scattering-power of Born's electron, *Proc. Roy. Irish Acad. Sect. A* **48**:91–122 (1942).
40. E. Schrödinger, A new exact solution in non-linear optics (two-wave system), *Proc. Roy. Irish Acad. Sect. A* **49**:59–66 (1943).
41. R. F. Streater and A. Wightman, *PCT, Spin and Statistics, and All That* (Princeton University Press, Princeton, 1964).
42. W. E. Thirring, *Classical Mathematical Physics: Dynamical Systems and Field Theory*, 3rd ed. (Springer-Verlag, New York, 1997).
43. R. Tumulka, Private communications (2004).
44. S. Weinberg, *The Quantum Theory of Fields. Vol. I, Foundations* (Cambridge University Press, Cambridge, 1995).
45. J. A. Wheeler and W. H. Zurek (eds.), *Quantum Theory and Measurement* (Princeton University Press, Princeton, 1982).
46. A. S. Wightman, Some comments on the quantum theory of measurement, in *Probabilistic Methods in Mathematical Physics*, Proceedings of the International Workshop held in Siena, May 6–11, 1991, F. Guerra, M. I. Loffredo, and C. Marchioro, eds. (World Scientific, River Edge, NJ, 1992), pp. 411–438.